Numerical Analysis of Generalized Semi-Markov Processes

Christoph Lindemann
Professor, Dr.

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University of Leipzig
Department of Computer Science
Augustusplatz 10
04109 Leipzig, Germany

Introduction

- Need for both stochastic and deterministic timing
    - system workloads are variable (i.e., stochastic)
    - system failures are stochastic
    - system activities are constant (i.e., are deterministic)
    - system repair times are constant

Introduction

- Research contribution
  - two theorems providing the foundation for an effective algorithmic generation of the transition kernel
  - **Theorem 1: Derivation of an algorithmic approach how kernel elements can always be computed by summation of transient state probabilities of continuous-time Markov chains**
  - **Theorem 2: Derivation of conditions on the building blocks of the GSMP under which kernel elements are constant**
    - algorithmic exploitation as key driver for fast methodology, increasing accuracy of solution and reducing memory requirements

Related Work

- results presented obtained in the years 1998 to 2001
  - i.e. concurrently to Kishor Trivedi's and his (former) PhD students' work, e.g.
    - **H. Choi, V.G. Kulkarni, and K.S. Trivedi, Markov Regenerative Stochastic Petri Nets, Performance Evaluation 1994**
    - **G. Ciardo and G. Li, Approximate transient analysis for subclasses of DSPN, Performance Evaluation 1999**
- Research contribution
  - efficient quantitative analysis of discrete-event systems with exponential and (concurrent) deterministic events
    - **numerical method for steady-state analysis**
    - **numerical method for transient analysis**
  - based on analysis of a GSSMC
  - **Lindemann, Shedler, Performance 96, Lindemann, Thümmler, Performance 99**

Results 1991 to 1995
GSMP without concurrent det. Events
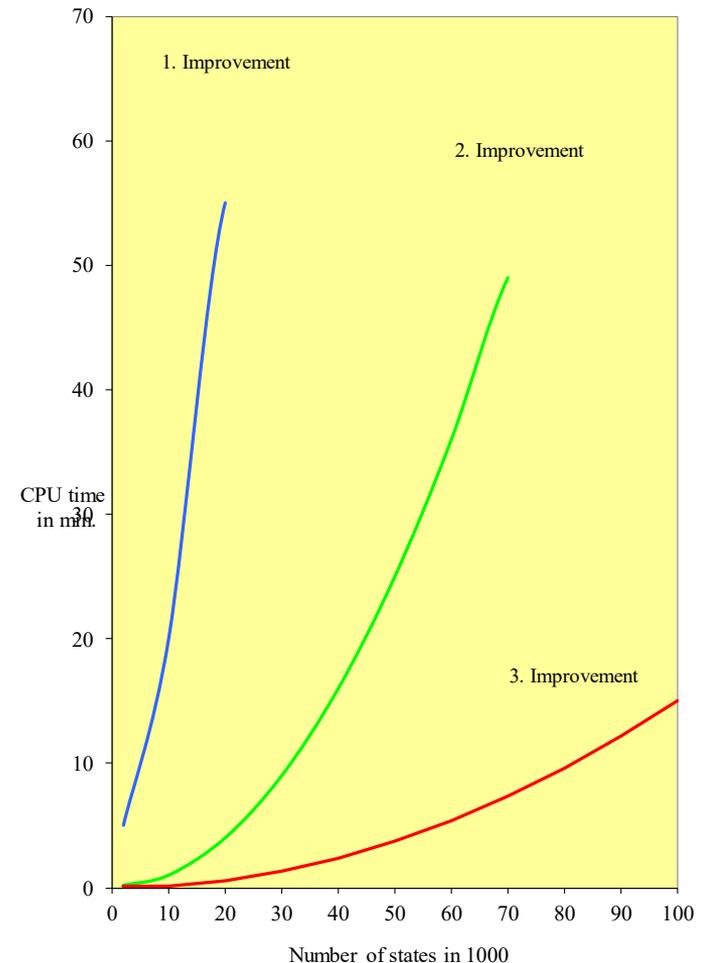
1. **improvement**

   uniformization method for computation of jump probabilities
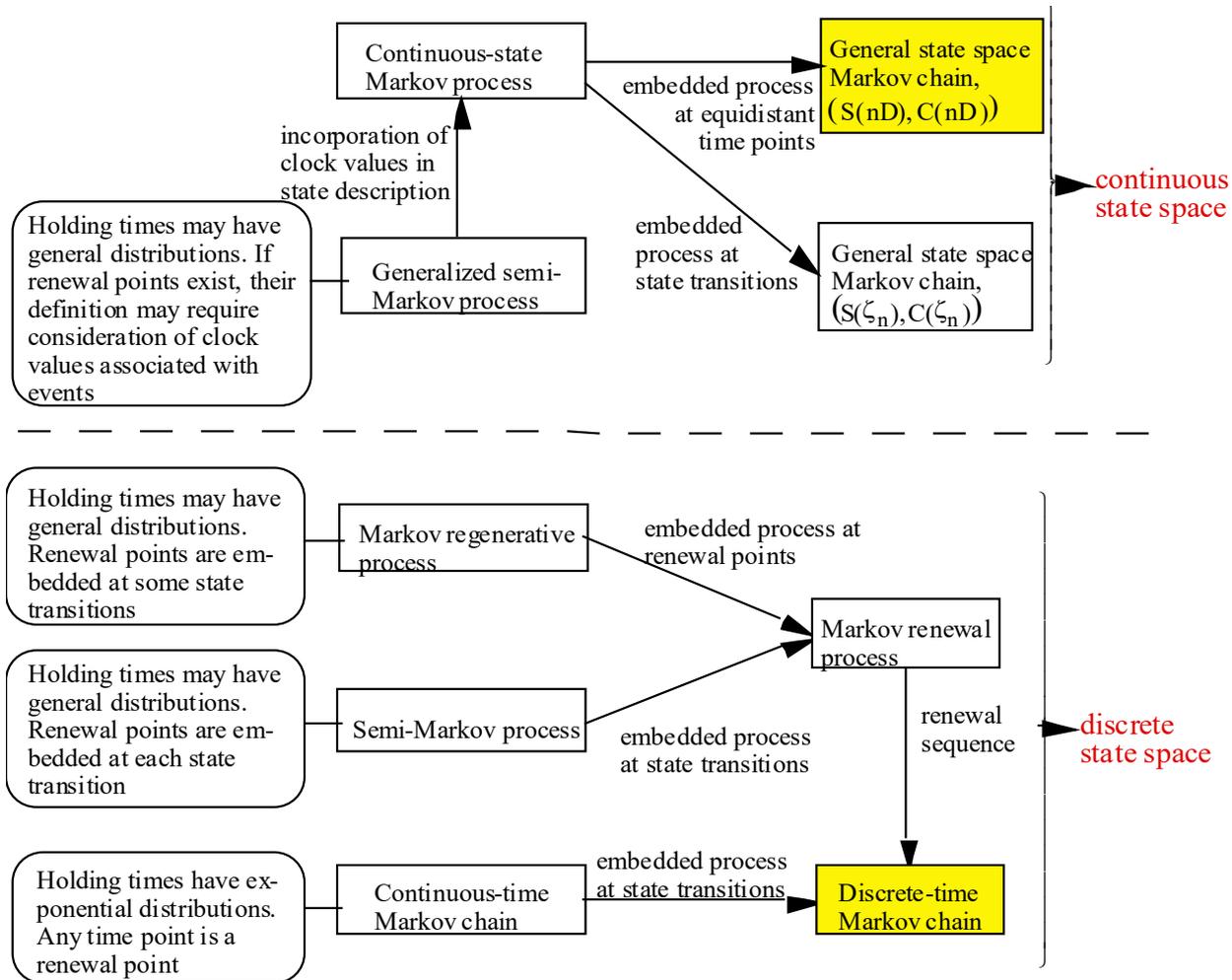
2. **improvement**

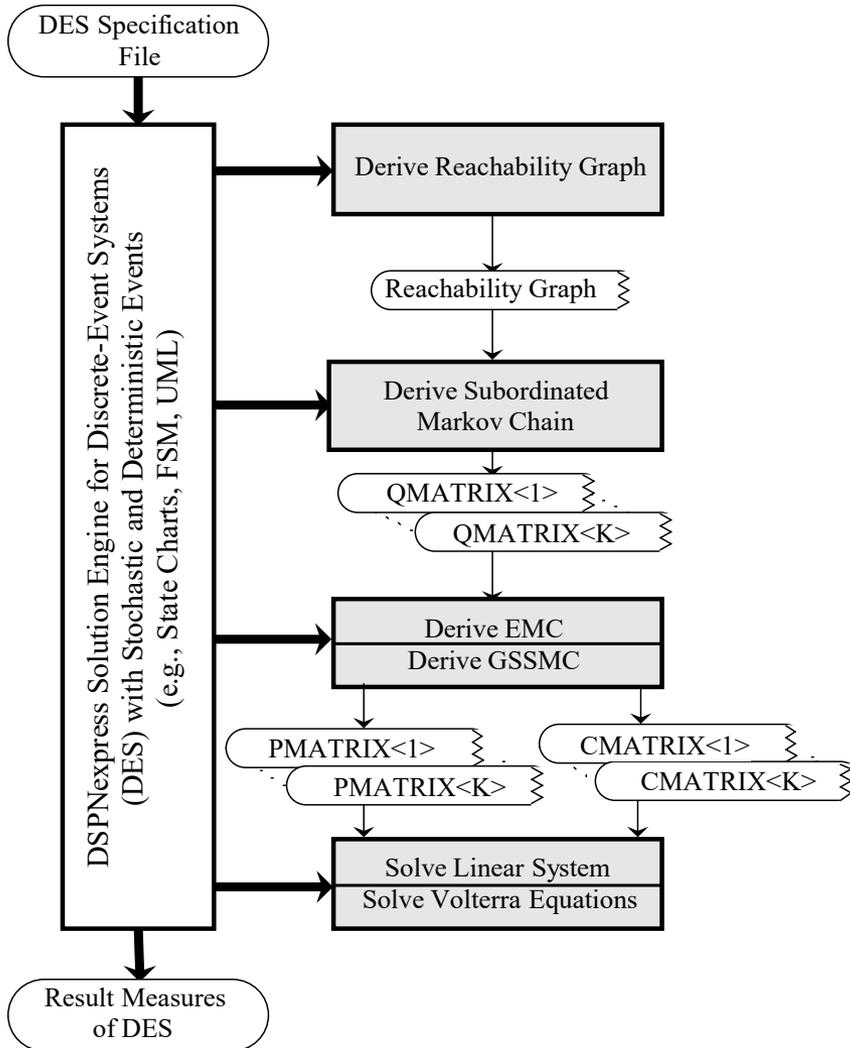   decomposition in subordinated Markov chains

3. **improvement**

   Reuse of already computed jump probabilities by exploitation of special structures and graph isomorphisms

Classification of Markov Prozesses

Continuous-state Markov process

embedded process at equidistant time points

General state space Markov chain, $(S(nD), C(nD))$

incorporation of clock values in state description

Holding times may have general distributions. If renewal points exist, their definition may require consideration of clock values associated with events

Generalized semi-Markov process

embedded process at state transitions

General state space Markov chain, $(S(\zeta_n), C(\zeta_n))$

continuous state space

Holding times may have general distributions. Renewal points are embedded at some state transitions

Markov regenerative process

embedded process at renewal points

Markov renewal process

Holding times may have general distributions. Renewal points are embedded at each state transition

Semi-Markov process

embedded process at state transitions

renewal sequence

discrete state space

Holding times have exponential distributions. Any time point is a renewal point

Continuous-time Markov chain

embedded process at state transitions

Discrete-time Markov chain

Outline of Numerical Analysis

(1) deriving state space

(2) determining the probability matrix P or the transition kernel $P(.)$

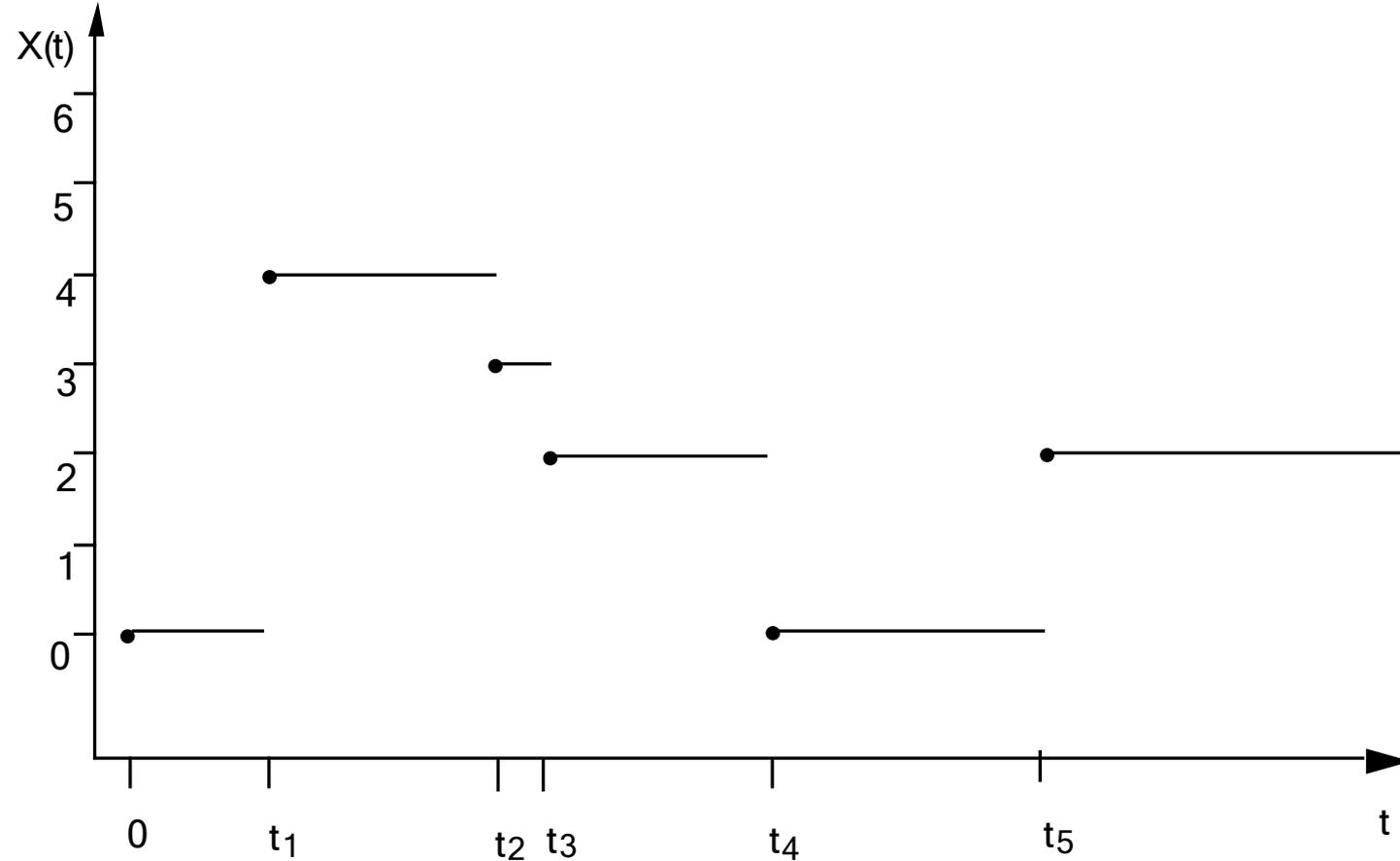(3) numerical solution phase

Sample Path of Stochastic Process

Illustration of 1. Embedding Technique

Discrete-time stochastic process (i.e., DTMC) embedded at state transitions (i.e., renewal points)
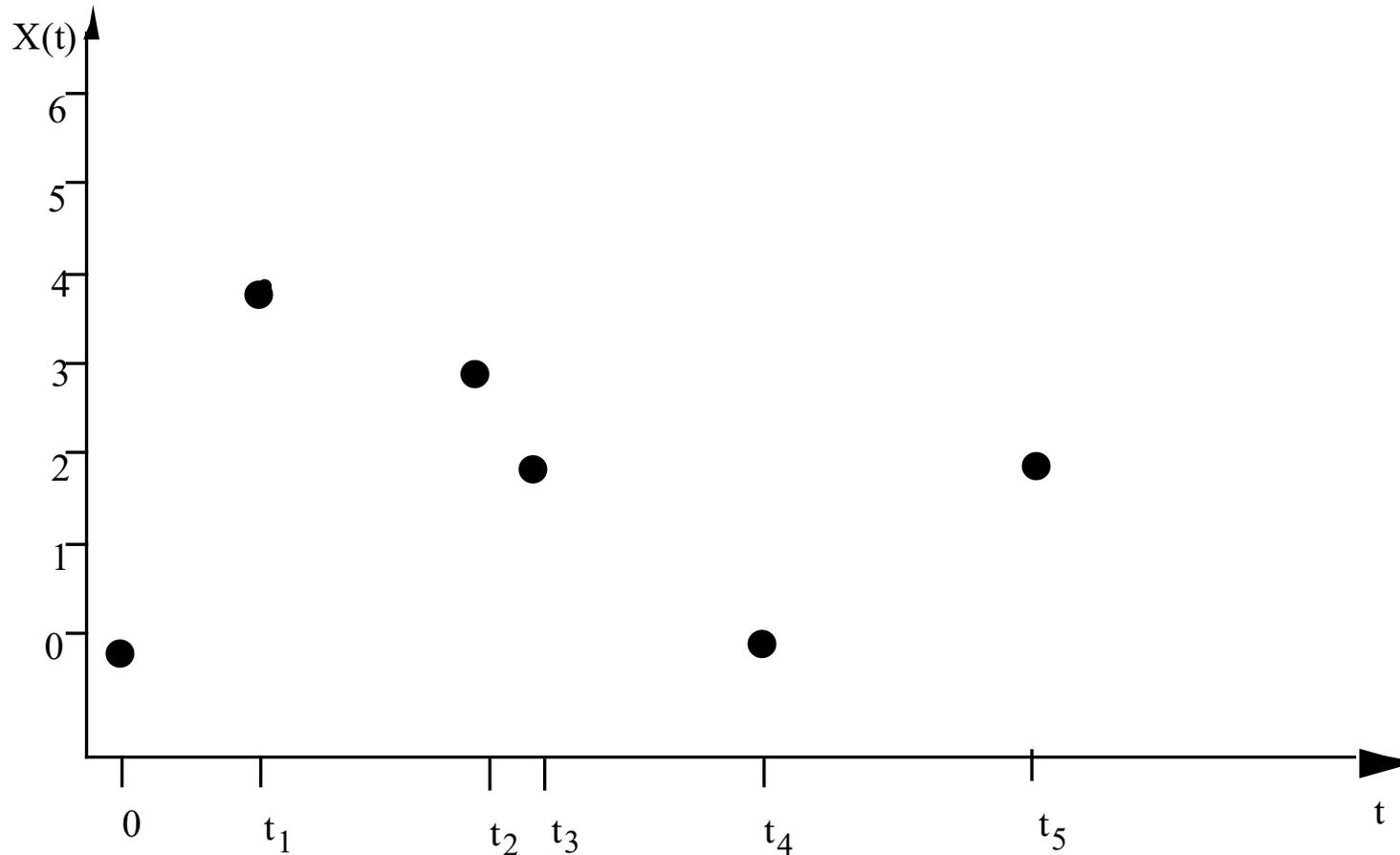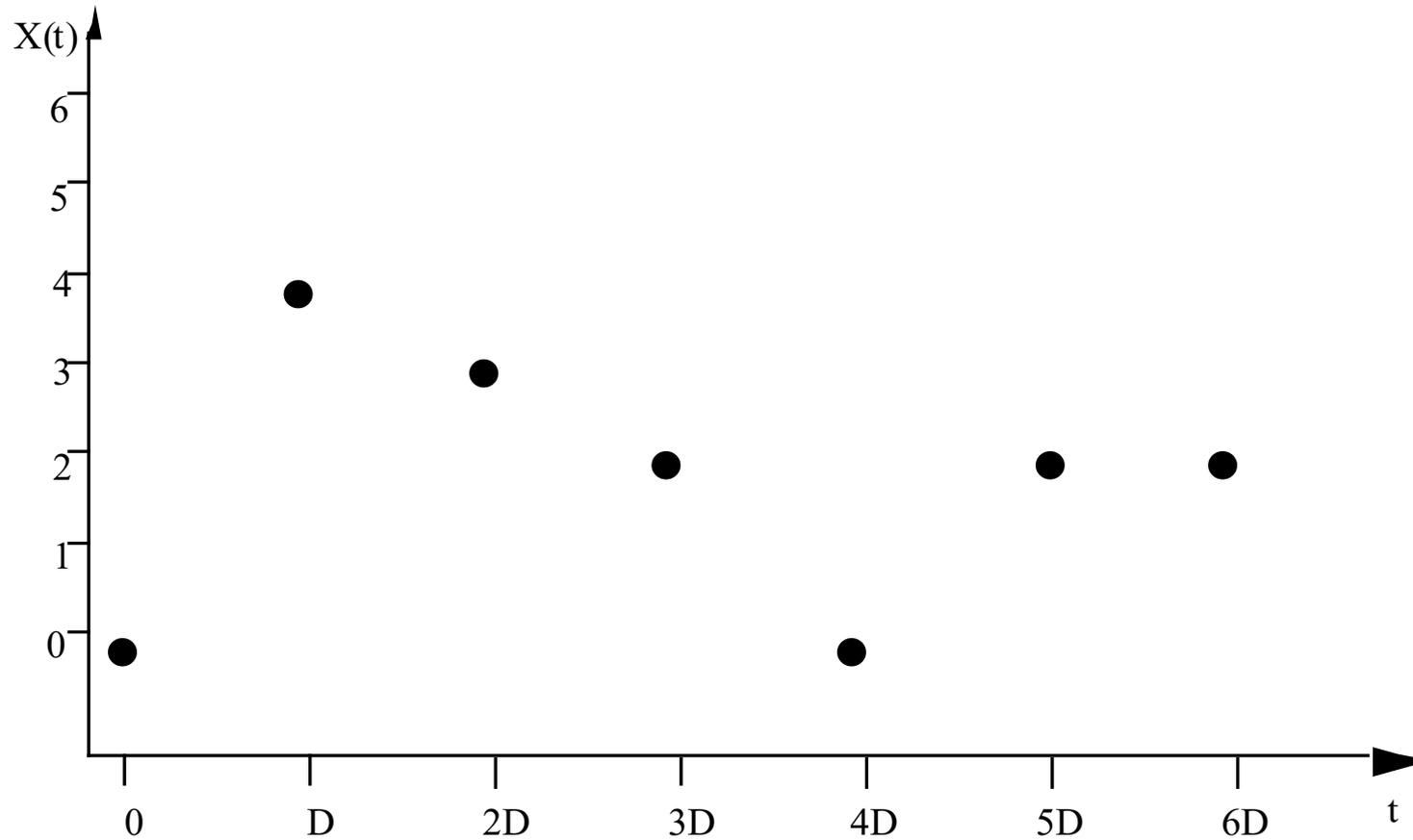
Illustration of 2. Embedding Technique

Discrete-time stochastic process (i.e., GSSMC) embedded at equidistant time points

General State Space Markov Chain (GSSMC)

- GSSMC is completely specified by
  - transition kernel $P(c_1, c_2, a_1, a_2)$
  - initial distribution at time $t = 0$
  - analysis leads to system of Volterra integral equations
- Key observations for efficiency of approach
  - transition kernel is separable
  - elements of transition kernel can be computed as sums of transient probabilities of CTMCs (subordinated Markov chains)
  - in general elements of transition kernel are continuous functions

GSSMC Transition Kernel (1)

- Vector of old clock readings c and set A for intervals of new clock readings
  - for two concurrent deterministic events

$$c = c(s_i) = \begin{cases} \emptyset & , s_i \in S_{exp} \\ c_1 & , s_i \in S_{det1} \\ (c_1, c_2) & , s_i \in S_{det2} \end{cases}$$

$$A = A(s_j) = \begin{cases} \emptyset & , s_j \in S_{exp} \\ (0, a_1] & , s_j \in S_{det1} \\ (0, a_1] \times (0, a_2] & , s_j \in S_{det1} \end{cases}$$

GSSMC Transition Kernel (2)

$$P(c_1, c_2, a_1, a_2) = \begin{pmatrix} P_{11} & P_{12}(a_1) & P_{13}(a_1, a_2) \\ P_{21}(c_1) & P_{22}(c_1, a_1) & P_{23}(c_1, a_1, a_2) \\ P_{31}(c_1, c_2) & P_{32}(c_1, c_2, a_1) & P_{33}(c_1, c_2, a_1, a_2) \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ N_1 \\ \hline N_1 + 1 \\ \vdots \\ N_1 + N_2 \\ \hline N_1 + N_2 + 1 \\ \vdots \\ N \end{matrix}$$

$$1 \quad \dots \quad N_1 \mid N_1 + 1 \dots N_1 + N_2 \mid N_1 + N_2 + 1 \quad \dots \quad N$$

Subordinated Markov Chain of s_i

- CTMC $\{X_i(t) : t \geq 0\}$ with state transitions corresponding to
  - occurrence of exponential events
  - state space $SMC_i = \{s \in S \mid s_i \xrightarrow{exp*} s\}$
  - initial distribution $P\{X_i(0) = s_i\} = 1$

- Probability for transition from state s_i to state s_j in time t via only exponential events is given by

$$P\big[s_i \xrightarrow{exp*} s_j\big]^{def} = P\big[X_i(t) = s_j \mid X_i(t) = s_i\big]$$

$$= 1_i^T \overset{t}{\cdot} e^{Q_i t} \cdot 1_j$$

Clock Reading Interval Decomposition

- GSMP is time-homogeneous
  - decomposing $(o, D]$ into
    - $(0, c_1]$
    - $(c_1, c_2]$
    - $(c_2, D]$

  $\rightarrow$ GSMP behaves like CTMC in these time intervals

  $\rightarrow$ kernel elements of the embedded GSSMC can be computed as summations of transient state probabilities of SMCs

Computation of Transition Kernel (1)

- Elements of the transition kernel of a GSSMC can always be determined by appropriate sums of transient state probabilities of continous-time Markov chains
  - assuming GSMP is at time nD in state s_i with det. Events $e_{l(i)}$ end $e_{m(i)}$ enabled.
  - → GSSMC resides in state (s_i, c_1, c_2) with $c_1 \leq c_2$ where c_1, c_2 are clock readings associated with events $e_{l(i)}$ and $e_{m(i)}$
  - state of GSMP at time $(n+1)D$ given the state at time nD is determined by
    - sequences of exp. events in the subintervals $((nD, nD + c_1], (nD + c_1, nD + c_2]$ and $(nD + c_2, (n+1)D]$
    - occurrence of the det. events $e_{l(i)}$ and $e_{m(i)}$ at instances of time $nD + c_1$ and $nD + c_2$

Computation of Transition Kernel (2)

Theorem 1

Let $\{X(t): t \geq 0\}$ be a finite-state GSMP with exponential and deterministic events.
Then all elements $p_{ij}(.)$ of transition kernel
$P(c_1, c_2, a_1, a_2)$ of the the embedded GSSMC
$\{X_n: n \geq 0\}$ can be computed simply by summation of transient state probabilities of continuous-time Markov chains.

Detecting constant Kernel Elements

Theorem 2

- Specifying building blocks of the GSMP under which kernel elements are constant

- Jump probabilities of GSMC are independent of clock readings

- Example: GSMPs underlying queueing systems
  - quasi birth-death arrival process
  - one or several deterministic servers
  - Infinite waiting room (i.e. MAP/D/c queues)
 - → **corresponding GSSMC is an ordinary DTMC**

Conditions for const. Kernel Elements

- Necessary condition for kernel elements of the GSSMC to be constant
  - clock readings of new deterministic events at time $(n + 1)D$ do not depend on the occurrence of exponential events in $[\,nD, (n + 1)D)$
  - for GSMPs with at most two deterministic events concurrently enabled
    - $p_{ij}(c_1, a_1) = p_{ij}(c_1)$
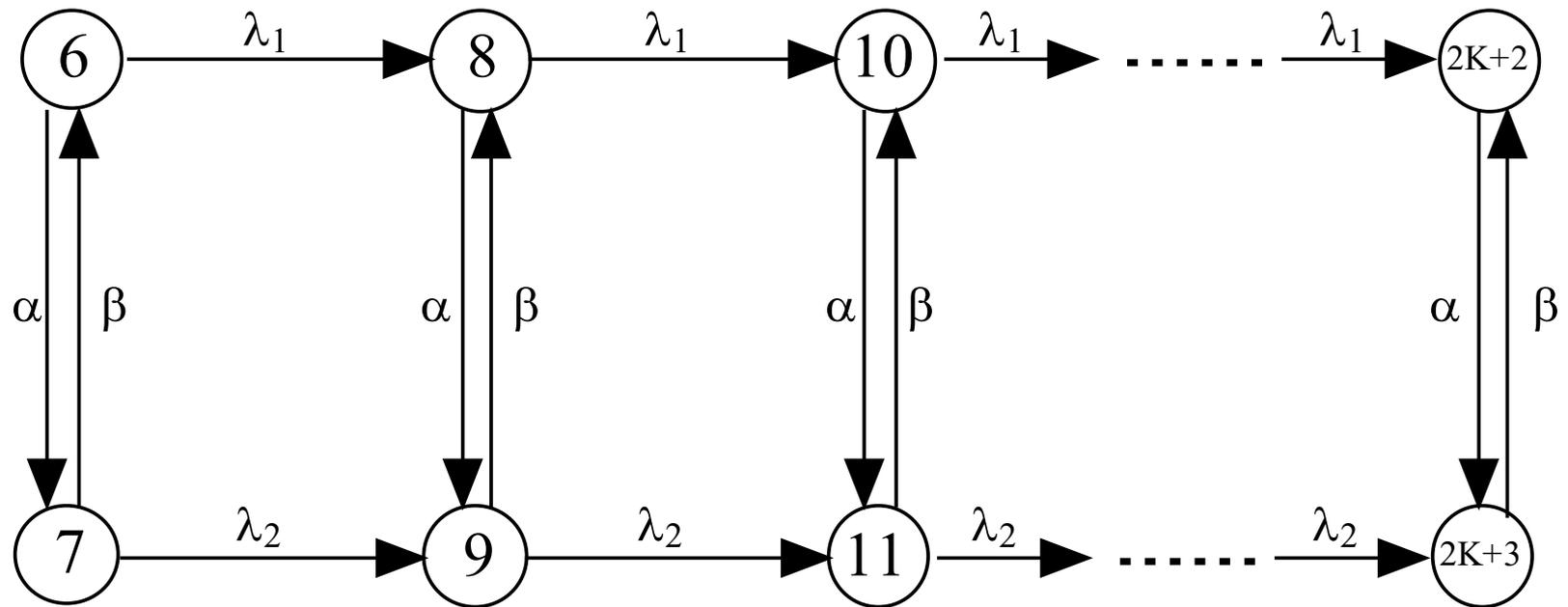   - $p_{ij}(c_1, c_2, a_1, a_2) = p_{ij}(c_1, c_2)$

Performance Results for MMPP/D/2/K

- Experimental setup 2002:
  - PC Workstation: Linux; 2,3 GHz processor; 2GB main memory
  - CPU times measured with *times*

# Reachability Graph of MMPP/D/2/K
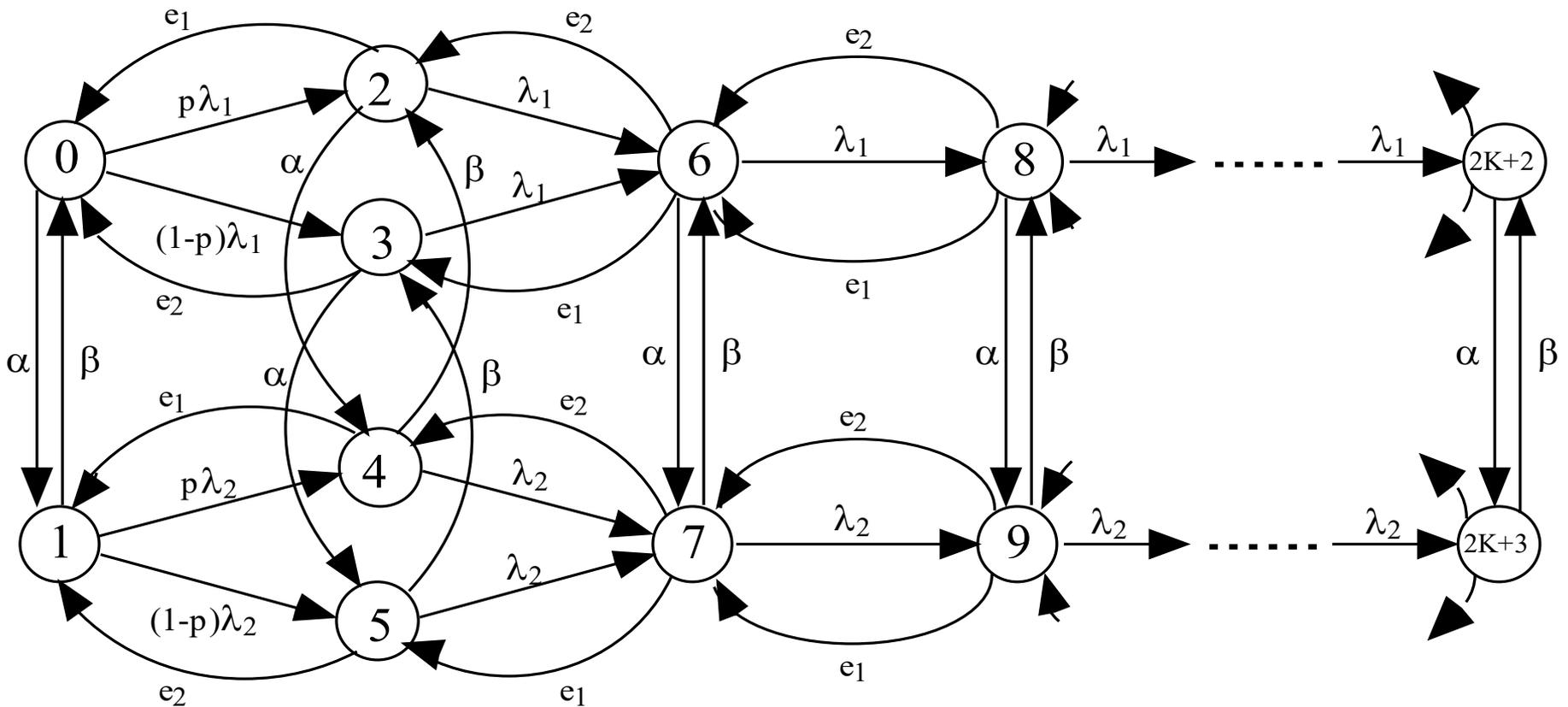
Subordinated CTMC of MMPP/D/2/K

Transition Kernel of MMPP/D/2/K
Classification of elements

| States of GSSMC	Nonzero entries	Constant entries	Functionals in 1 variable	Functionals in 2 variables	Functionals in 3 variables	Functionals in 4 variables
2004	2004997	99,30 %	0,20 %	0,49 %	$2,5 \cdot 10^{-4}$ %	$1,0 \cdot 10^{-4}$ %
4008	8009997	99,65 %	0,10 %	0,25 %	$6,3 \cdot 10^{-5}$ %	$2,5 \cdot 10^{-5}$ %
6012	18014997	99,77 %	0,07 %	0,17 %	$2,8 \cdot 10^{-5}$ %	$1,1 \cdot 10^{-5}$ %
8016	32019997	99,83 %	0,05 %	0,12 %	$1,6 \cdot 10^{-5}$ %	$6,3 \cdot 10^{-6}$ %
10020	50024997	99,86 %	0,04 %	0,10 %	$1,0 \cdot 10^{-5}$ %	$4,0 \cdot 10^{-6}$ %
12024	72029997	99,88 %	0,03 %	0,08 %	$6,9 \cdot 10^{-6}$ %	$2,8 \cdot 10^{-6}$ %
14028	98034997	99,90 %	0,03 %	0,07 %	$5,1 \cdot 10^{-6}$ %	$2,0 \cdot 10^{-6}$ %
16032	128039997	99,91 %	0,02 %	0,06 %	$3,9 \cdot 10^{-6}$ %	$1,7 \cdot 10^{-6}$ %
18036	162044997	99,92 %	0,02 %	0,06 %	$3,1 \cdot 10^{-6}$ %	$1,2 \cdot 10^{-6}$ %
20040	200049997	99,93 %	0,02 %	0,05 %	$2,5 \cdot 10^{-6}$ %	$1,0 \cdot 10^{-6}$ %

Loss Probability vs. Burstfactor
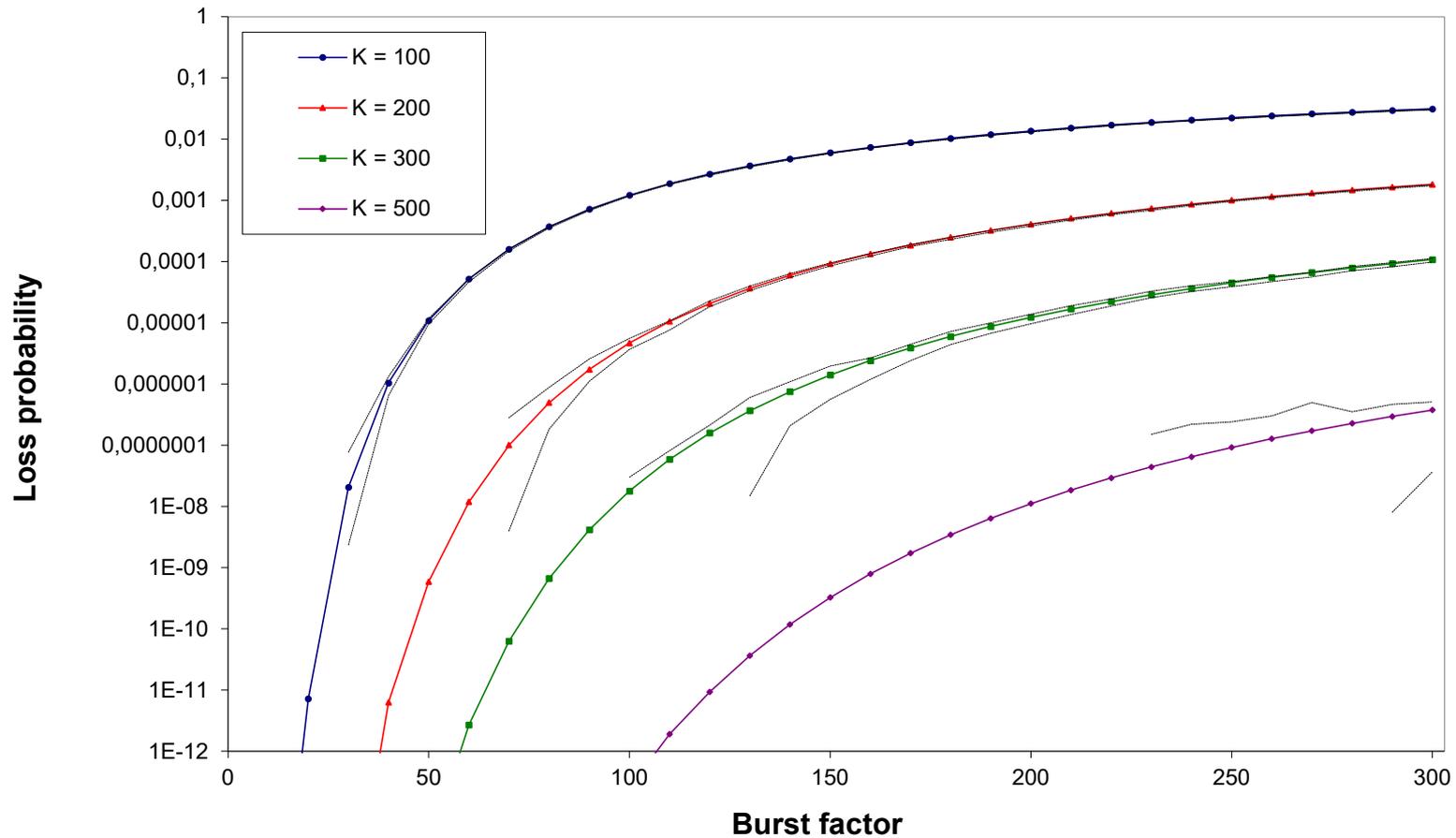
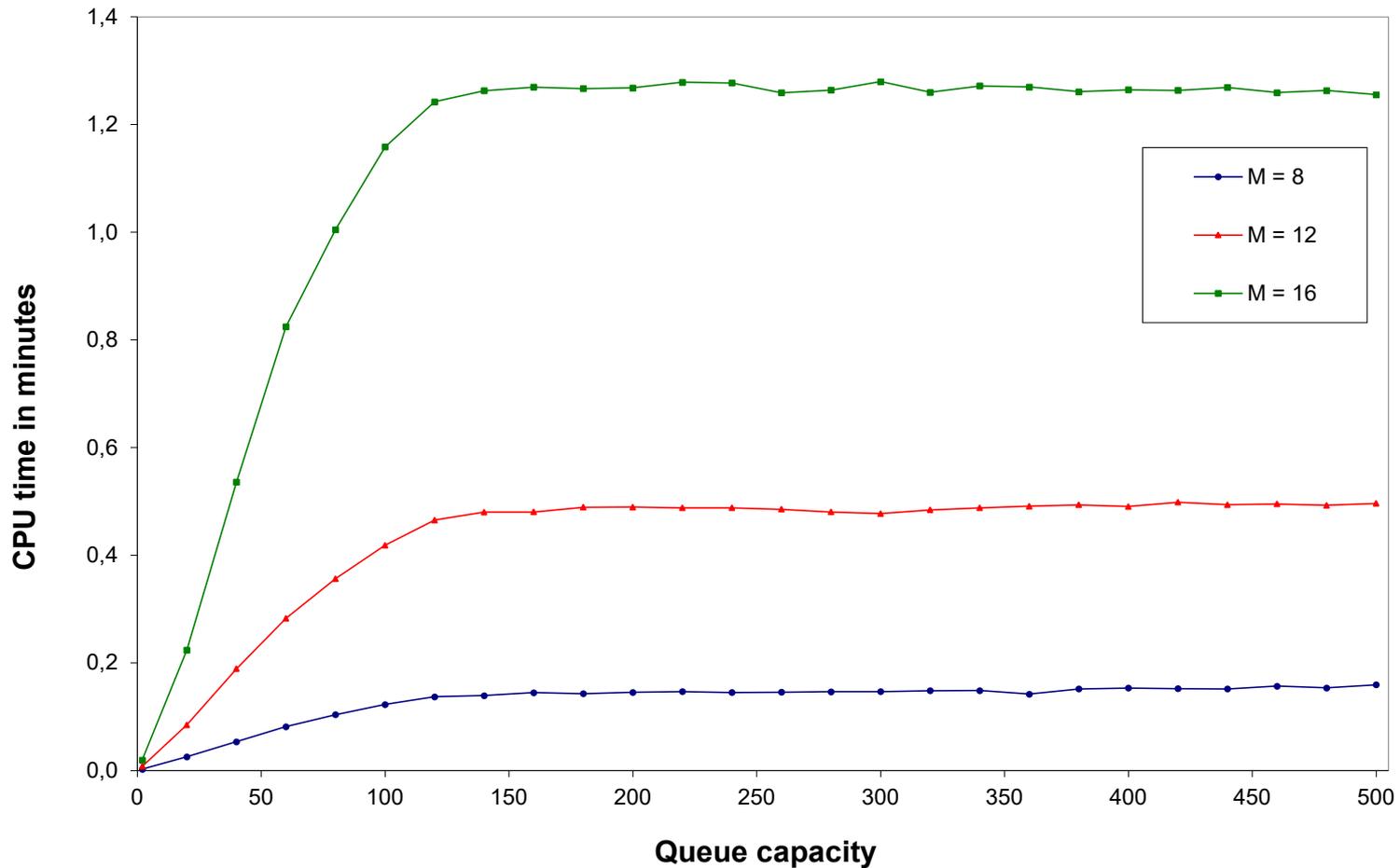# Loss Probability vs. Queue Capacity
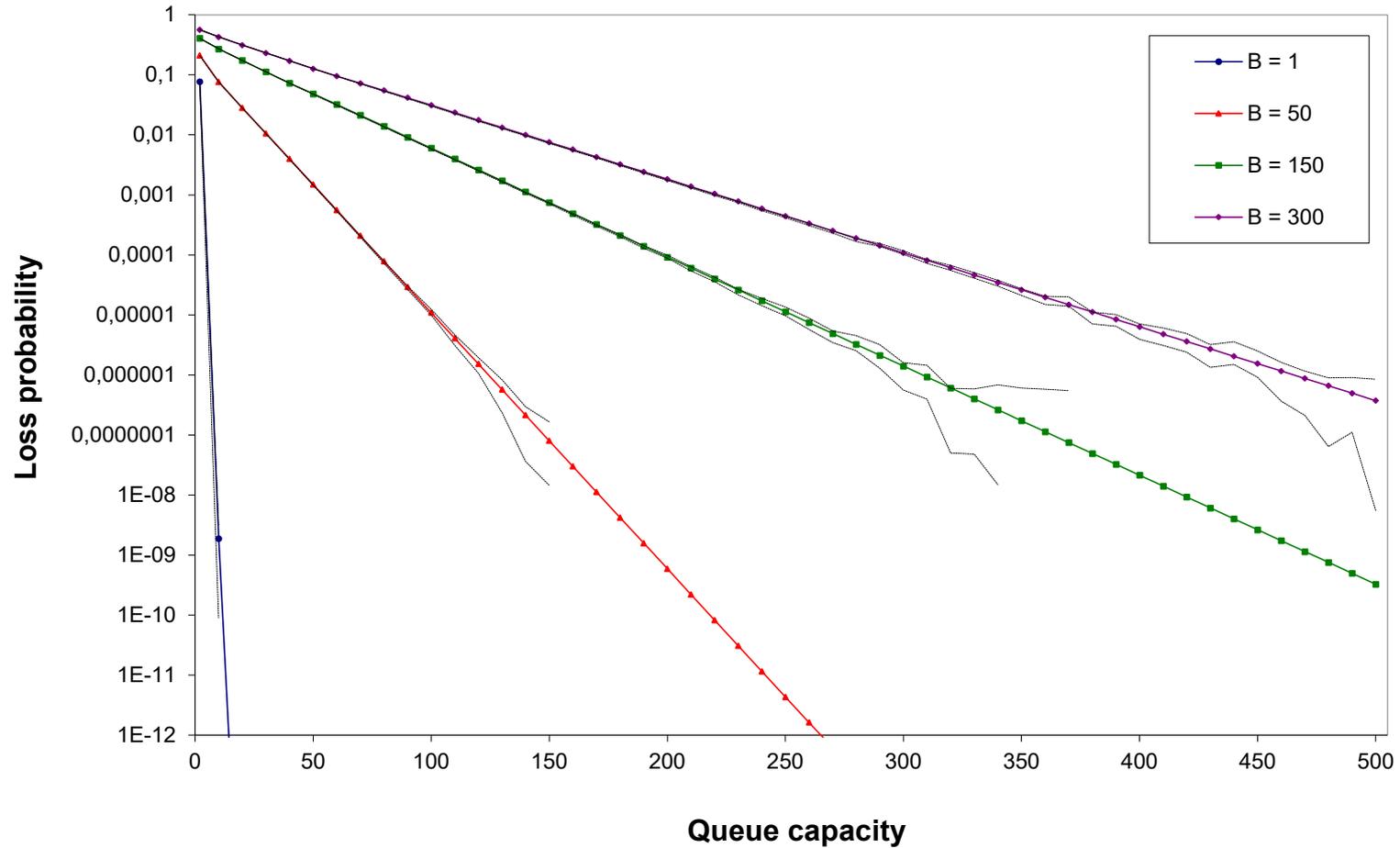
Transient and Steady-State Analysis
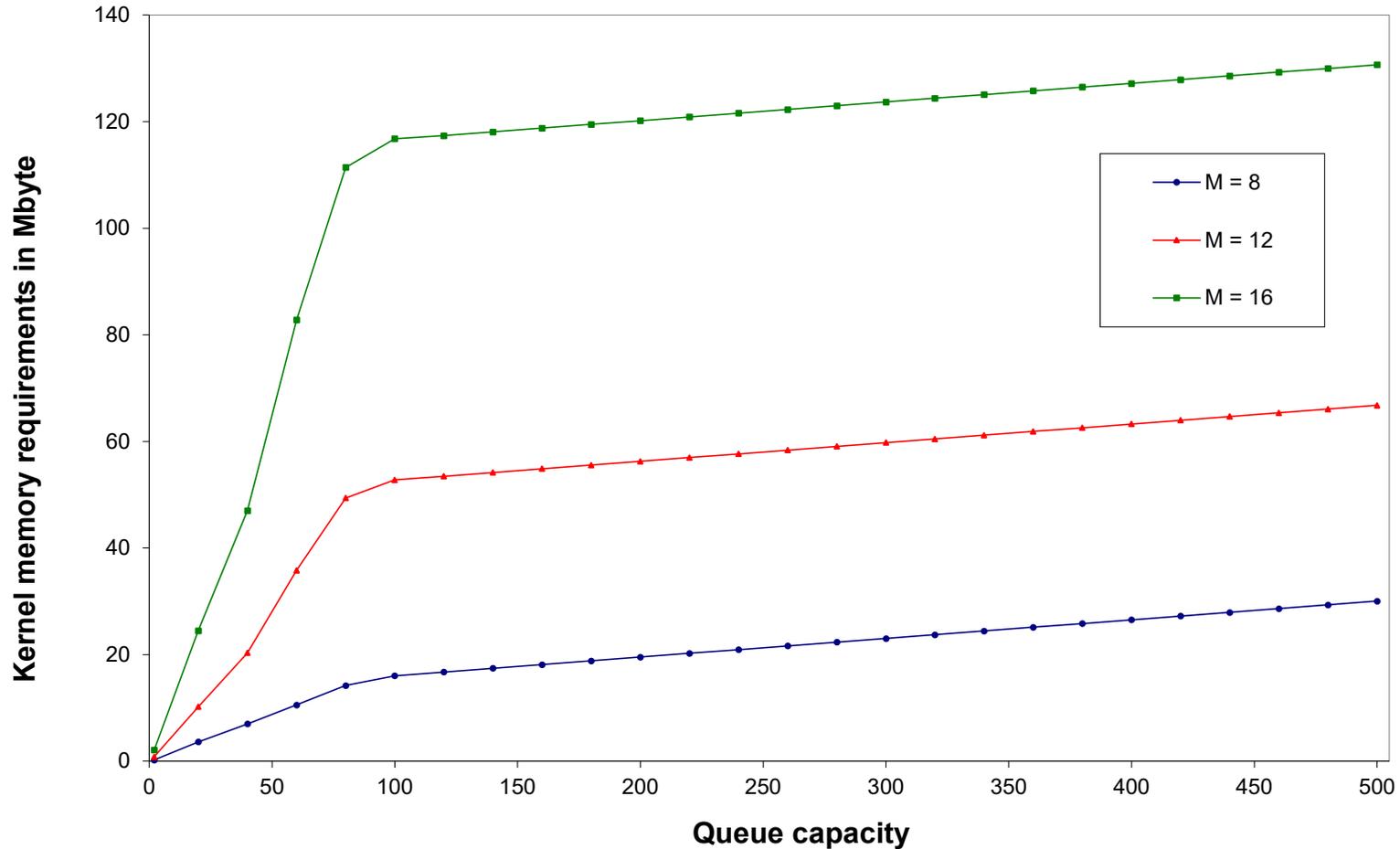## Computational Effort
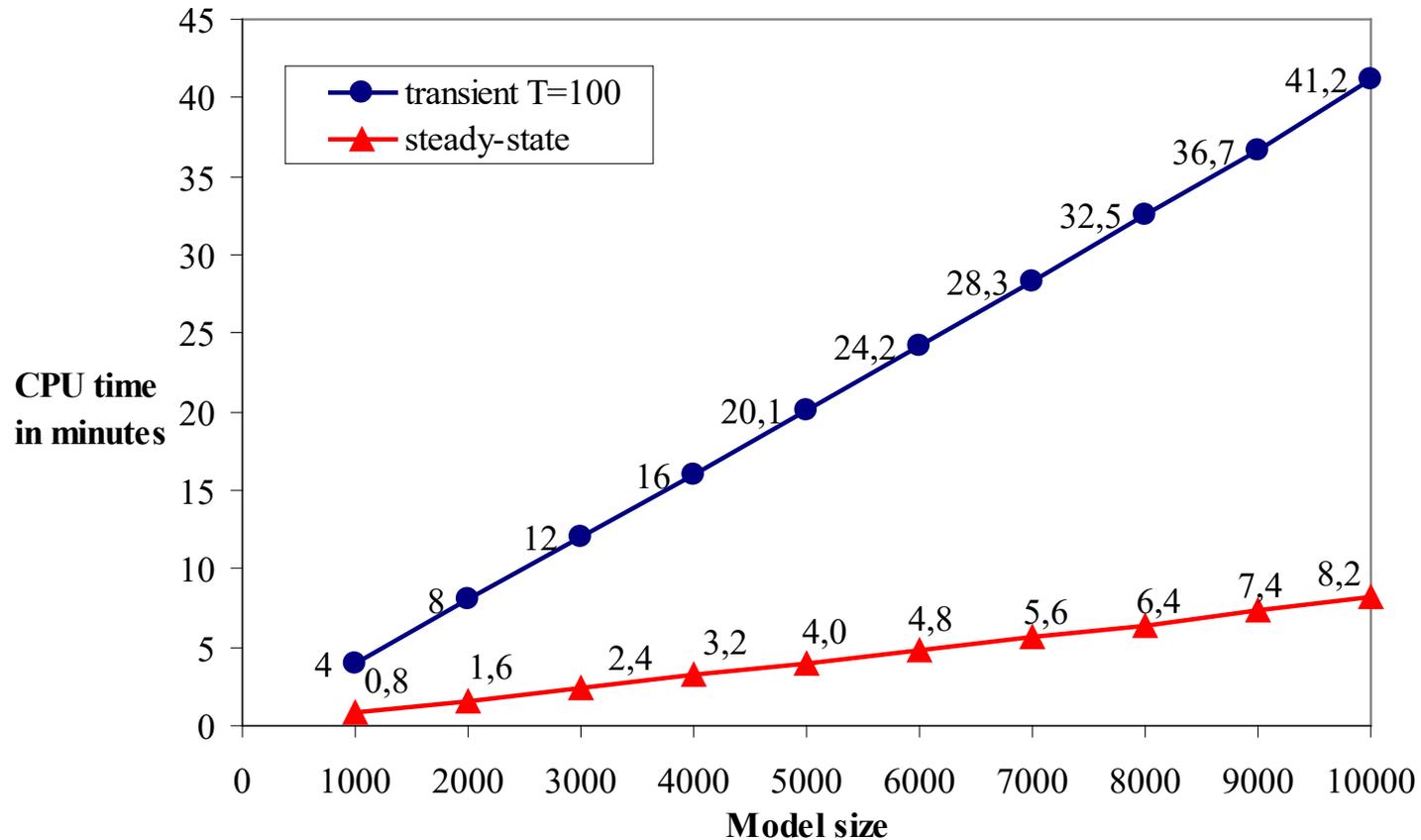
Generation of transition kernel
CPU time

Generation of Transition Kernel
Memory Requirements

Conclusions

- Two theorems providing the foundation for an effective algorithmic generation of the transition kernel

- **Theorem 1: Derivation of an algorithmic approach how kernel elements can always be computed by summation of transient state probabilities of continuous-time Markov chains**

- **Theorem 2: Derivation of conditions on the building blocks of the GSMP under which kernel elements are constant**

- Algorithmic exploitation is key driver for fast solver

- Increases accuracy of solution and reduces memory requirements