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Numerical analysis of deterministic and stochastic Petri nets with concurrent deterministic transitions

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Abstract

This paper introduces an efficient numerical algorithm for the steady-state analysis of deterministic and stochastic Petri nets (DSPNs) without structural restrictions on the enabling of deterministic transitions. The method rests on observation, at equidistant time points, of the continuous-time Markov process that records tangible markings of the DSPN and remaining firing times associated with deterministic transitions. This approach results in the analysis of a general state space Markov chain whose system of stationary equations can be transformed into a system of Volterra equations. The techniques of this paper are also applicable to queueing networks, stochastic process algebras, and other discrete-event stochastic systems with an underlying stochastic process which can be represented as a generalized semi-Markov process with exponential and deterministic events.

Keywords: Stochastic Petri nets; General state space Markov chains; Numerical transient analysis of continuous-time Markov chains; Numerical solution of Volterra integral equations

1. Introduction

Deterministic and stochastic Petri nets (DSPNs) are a stochastic modeling formalism with graphical representation which include both exponentially distributed and deterministic delays. Under the restriction that in any marking of a DSPN at most one deterministic transition is enabled and timed transitions have the execution policy "race with enabling memory" [1], Ajmone Marsan and Chiola introduced a numerical method for computing the steady-state solution of a DSPN using an embedded Markov chain [2]. More recently, Choi, Kulkarni, and Trivedi have observed that the marking process of such DSPNs is a Markov regenerative stochastic process and showed how to obtain transient and stationary distributions [4]. Under the same restriction, Telek and Bobbio showed that the marking process of a DSPN in which timed transitions have the execution policy "race with age memory" [1] can also be represented as a Markov regenerative stochastic process [17]. Several efforts aiming at development of

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numerical methods for steady-state analysis of DSPNs with concurrently enabled deterministic transitions have been reported. Ciardo, German and Lindemann have shown that DSPNs with concurrently enabled deterministic transitions can be analyzed by a Markov regenerative process, provided that the concurrently enabled deterministic transitions fire at the same time [6]. To relax this restriction, German and Lindemann have proposed the use of supplementary variables [7] that incorporate the elapsed firing times of deterministic transitions into a state description of a DSPN and analysis of a continuous-time Markov process with continuous state space [10]. However, the practical applicability of this approach is limited because it requires numerical solution of a system of partial differential equations with complicated boundary value functions.

In this paper, we introduce an efficient numerical method for the stationary analysis of DSPNs with concurrently enabled deterministic transitions. The numerical method is based on the analysis of an appropriately embedded general state space Markov chain (GSSMC), rather than analysis of the continuous-time Markov process as in the method of supplementary variables. The GSSMC is defined by observation, at equidistant time points, of the continuous time Markov process that records tangible markings of the DSPN and remaining firing times associated with deterministic transitions. This embedded GSSMC and the continuous-time Markov process have the same limiting distribution, provided that limits for the Markov process exist.

The stationary equations of the embedded GSSMC comprise a system of integro-differential equations with constant boundary conditions. Using integration by parts, this system of integro-differential equations can be transformed into a system of Volterra integral equations of the second type, for which standard numerical solution methods are available [3]. Numerical methods for such systems are considerably simpler than any numerical method for solving a system of partial differential equations as required by the approach of supplementary variables [10]. The Volterra integral equations contain entries of the transition kernel of the GSSMC (heuristically, that is a family of probability matrices) which specify probabilities of its state transitions as functions of clock readings. We illustrate how for fixed values of clock readings the entries of the transition kernel can be numerically determined by extending the concept of subordinated Markov chains [14]. As a consequence, the proposed numerical method based on analysis of the embedded GSSMC has a number of advantages over the method of supplementary variables [7]. The numerical method introduced in this paper is also applicable to queueing networks, stochastic process algebras, and other discrete-event stochastic systems with an underlying stochastic process which can be represented as a generalized semi-Markov process with exponential and deterministic events.

The remainder of this paper is organized as follows. In Section 2 we show how to define the general state space Markov chain for underlying a DSPN with concurrent deterministic transitions and outline the numerical computation of its transition kernel. The derivation of the stationary equations is presented in Section 3. To illustrate the proposed numerical method, in Section 4 we apply the method for steady-state analysis of a DSPN of a M/D/2/K queueing system. Finally, concluding remarks are given.

2. The approach based on a general state space Markov chain

2.1. The marking process of a deterministic and stochastic Petri net

Formally, a Petri net is a directed bipartite graph with one set of vertices called *places* (drawn as circles) and the other called *transitions* (drawn as bars). Places may contain *tokens* which are drawn as dots. Places and transitions are connected by directed arcs or inhibitor arcs (drawn with an circled head).

Arcs may be labelled with integer numbers denoting their multiplicity. The default multiplicity of an arc is one. A transition is said to be *enabled*, if all of its input places contain at least as many tokens as the multiplicity of the corresponding input arc and all of its inhibitor places contain less tokens than the multiplicity of the corresponding inhibitor arc. A transition *fires* by removing from each input place as many tokens as the multiplicity of the corresponding input arc, and by adding to each output place as many tokens as the multiplicity of the corresponding output arc. In deterministic and stochastic Petri Nets (DSPNs [2]) three types of transitions exist: immediate transitions drawn as thin bars fire without delay, exponential transitions drawn as empty bars fire after an exponentially distributed delay whereas deterministic transitions drawn as black bars fire after a constant delay.

The numerical analysis of DSPNs proceeds by computing transient or stationary distributions for its underlying continuous-time stochastic process $\{S(t):\ t \geq 0\}$, which is denoted as the *marking process* of the DSPN. The state space of the marking process is defined by tangible markings and its state transition diagram is given by the tangible reachability graph of the DSPN. Since the deterministic distribution does not show absence of memory, a proper definition of the stochastic behavior of DSPNs requires the specification how the selection of the next transition to fire is performed and how the model keeps track on past history. Such formal specifications for the semantics of transition firings in DSPNs have been introduced in [1]. Such a set of specifications has been called an *execution policy* and specifies the method for computing the remaining firing time of timed transitions after a marking change. Throughout this paper, we assume that among all enabled timed transitions in a DSPN the one with the minimum remaining firing time determines the next marking change. Furthermore, after a marking change each timed transition newly enabled samples a remaining firing time from its firing delay distribution and each timed transition, which has already been enabled in the previous marking and is still enabled in the current marking, keep its remaining firing time. This stochastic behavior corresponds to the execution policy *race with enabling memory* as defined in [1] and also coincidences with the state transition mechanism in a generalized semi-Markov process [11].

We allow exponential transitions to have a marking-dependent firing delays whereas firing delays of deterministic transitions have to be fixed. Furthermore, we assume that the reachability graph of the DSPN comprises of a finite number of markings and we define the set S of tangible markings by

$$S = \{1, 2, \ldots, N\}.$$

Following Haas and Shedler [13], the marking process of such DSPNs can be represented as a finite-state generalized semi-Markov process (GSMP) with exponential and deterministic events. A GSMP is a continuous time stochastic process $\{S(t):\ t \geq 0\}$ that records the tangible marking of the DSPN as it evolves over time. Thus, tangible markings of a DSPN correspond to states of the GSMP. Exponential and deterministic transitions of the DSPN define exponential events and deterministic events of the GSMP, respectively. Timed transitions enabled in a tangible marking of a DSPN correspond to the set of active events, associated with a state of the GSMP. Preempting the firing of a timed transition can easily be represented in the state transition mechanism of a GSMP by cancelling the corresponding event. Compound transition probabilities specifying the probability of firing sequences of immediate transitions after the firing of a timed transition of DSPN are represented by state transition probabilities of the GSMP valid after the occurrence of the event corresponding to this timed transition. To simplify the computation of these compound transition probabilities, we restrict the discussion to DSPNs for which firing weights associated with immediate transitions can be specified on the net level [5], i.e., the DSPN is confusion-free. Marking-dependency by a scaling factor as defined in [1] can be represented in a GSMP by state-dependent definitions of speeds associated with events.

2.2. The embedded general state space Markov chain

Enumerate the deterministic transitions of the DSPN by $t_1, t_2, \ldots, t_M$ and define D_m to be the firing delay of transition t_m $(1 \leq m \leq M)$. Let $C_m(t)$ be the remaining firing time associated with deterministic transition t_m at time t. In any state in which deterministic transition t_m is not enabled, we define $C_m(t) = 0$. Then, using the method of supplementary variables [7], we can derive a continuous-time Markov process $\{X(t)\colon\ t \geq 0\}$ with general (continuous) state space

$$X(t) = (S(t), C_1(t), C_2(t), \ldots, C_M(t)) .$$

Unfortunately, when deterministic transitions are concurrently enabled, the practical applicability of this approach is limited because the stationary analysis requires numerical solution of a system of partial differential equations with complicated boundary value functions [10]. Therefore, we consider a discrete-time stochastic process $\{X(nD)\colon\ n \geq 0\}$ by observing of the process $\{X(t)\colon\ t \geq 0\}$ at a sequence $\{nD\colon\ n \geq 0\}$ of fixed times for some appropriately defined step size $D > 0$

$$X(nD) = (S(nD), C_1(nD), C_2(nD), \ldots, C_M(nD)) . \tag{1}$$

Heuristically, $S(nD)$ represents the state (tangible marking of the DSPN) and $C_m(nD)$ represents the mth component of the clock-reading vector (remaining firing time of deterministic transitions t_m) at time nD. The memoryless property of the exponential distribution implies that $\{X(nD)\colon\ n \geq 0\}$ is a GSSMC, i.e., it satisfies the Markov property. If $\{X(nD)\colon\ n \geq 0\}$ is an aperiodic, positive recurrent chain with a regeneration set; that is, $\{X(nD)\colon\ n \geq 0\}$ is a Harris ergodic chain [9], the discrete-time process $\{X(nD)\colon\ n \geq 0\}$ has a unique stationary distribution. Otherwise, the presented approach allows the computation of time-averaged distributions. Using such a "stochastic skeleton approach", for the case that a stationary distribution exists, we have

$$\lim_{t\to\infty} P\{X(t) \in A\} = \lim_{n\to\infty} P\{X(nD) \in A\} \tag{2}$$

for any appropriate (i.e., "measurable") set A. In particular, by considering the sets $A = \{i\} \times [0, D_1] \times [0, D_2] \times \cdots \times [0, D_M]$ for $1 \leq i \leq N$, we can conclude that the stationary or time-averaged distribution of the discrete-time process $\{S(nD)\colon\ n \geq 0\}$ is equal to the stationary or time-averaged distribution of the continuous-time process $\{S(t)\colon\ t \geq 0\}$ which represents the marking process of the DSPN. If a stationary distribution exists, that is

$$\lim_{t\to\infty} P\{S(t) = i\} = \lim_{n\to\infty} P\{S(nD) = i\} \quad \text{for } 1 \leq i \leq N. \tag{3}$$

Using Eq. (3), we show that the stationary analysis of the marking process $\{S(t)\colon\ t \geq 0\}$ requires only numerical transient analysis of appropriately defined continuous-time Markov chains and the numerical solution of a system of Volterra integral equations of the second type. As shown in [3], numerical methods for such systems are considerably simpler than any numerical method for solving a system of partial differential equations as required by the approach of supplementary variables [10].

To determine the value for the step size D, so that stationary analysis of the GSSMC is numerically as simple as possible (i.e., all entries of its transition kernel can be computed using numerical transient analysis of continuous-time Markov chains and do not require transient analysis of more general stochastic processes), we observe for $n \geq 0$

(i) when deterministic transitions have the same firing delay (i.e., $D_1 = D_2 = \cdots = D_M$) all deterministic transitions which are already enabled at time nD_1 fire or get preempted before time $(n+1)D_1$.

(ii) when deterministic transitions have the same firing delay, deterministic transitions that become newly enabled in the interval $(nD_1, (n+1)D_1]$ cannot fire prior to time $(n+1)D_1$.

Thus, for this case, we define the time step D of the GSSMC as $D = D_1$. Note that, in general, the discrete-state stochastic process $\{S(nD)\colon\ n \geq 0\}$ does not satisfy the Markov property, i.e., is not a discrete-time Markov chain. However, for some interesting special cases this process is indeed a discrete-time Markov chain (see e.g., the stationary analysis of the M/D/k queueing system for any $k \geq 1$ [8]). Moreover, for a large gamut of cases, the discrete-state stochastic process $\{S(nD)\colon\ n \geq 0\}$ is almost a discrete-time Markov chain since most of the entries in the transition kernel of the GSSMC are constants (i.e., independent from the remaining firing times of deterministic transitions).

Now, let us consider the case that deterministic transitions have different firing delays. As shown in [16], by defining the time step D of the GSSMC as

$$D = \min\{D_1, D_2, \ldots, D_M\} \tag{4}$$

all entries of its transition kernel can also be computed using numerical transient analysis of continuous-time Markov chains. To provide a unified scale for the clock readings for deterministic transitions, for $1 \leq m \leq M$ and $n \geq 0$, we define:

- the scaling factor α_m associated with deterministic transition t_m given by $\alpha_m = \lfloor D_m / D \rfloor$,
- the lower subinterval $(nD, nD + \delta_m]$ associated with deterministic transition t_m of the interval $(nD, (n+1)D]$, where $\delta_m = (\alpha_m + 1)D - D_m$,
- the upper subinterval $(nD + \delta_m, (n+1)D]$ associated with deterministic transition t_m of the interval $(nD, (n+1)D]$.

We denote the length of the upper subinterval $(nD + \delta_m, (n+1)D]$ by γ_m that is,

$$\gamma_m = D - \delta_m = D_m - \alpha_m D.$$

According to these definitions, $\delta_m + \gamma_m = D$ for $1 \leq m \leq M$. If D_m is an integer multiple of D that is, $D_m = \alpha_m D$, then $\delta_m = D$ and $\gamma_m = 0$. Now, we observe

(iii) deterministic transitions that become newly enabled in the interval $(nD, (n+1)D]$ cannot fire prior to time $(n+1)D$.

(iv) when a deterministic transition t_m becomes enabled in the lower subinterval $(nD, nD + \delta_m]$, its remaining firing time at time $(n+1)D$ must lie in the interval $((\alpha_m - 1)D + \gamma_m, \alpha_m D]$. Consequently, deterministic transition t_m fires α_m intervals later. That is during the interval $((n+\alpha_m-1)D, (n+\alpha_m)D]$.

(v) when a deterministic transition t_m becomes enabled in the upper subinterval $(nD + \delta_m, (n+1)D]$, its remaining firing time at time $(n+1)D$ must lie in the interval $(\alpha_m D, \alpha_m D + \gamma_m]$. Consequently, deterministic transition t_m fires $\alpha_m + 1$ intervals later. That is during the interval $((n+\alpha_m)D, (n+\alpha_m+1)D]$.

From the definitions of the lower and upper subinterval follows that if D_m is an integer multiple of D (that is, $D_m = \alpha_m D$), deterministic transition t_m fires in the interval $((n+\alpha_m - 1)D, (n+\alpha_m)D]$ irrespective of when it became enabled in the interval $(nD, (n+1)D]$.

2.3. The transition kernel

The stationary analysis of the GSSMC $\{X(nD)\colon\ n \geq 0\}$ is based on its transition kernel which specifies probabilities for state transitions in the GSSMC. Recall that we assume that the marking process underlying the DSPN consists of N tangible markings (subsequently denoted as states). For ease of exposition, we restrict the discussion to DSPNs in which at most two deterministic transition may be concurrently enabled. Then, the subset of states in which only exponential transitions are enabled is denoted by S_{exp}. Similarly, the subsets of states in which one deterministic transition and two deterministic transitions are (concurrently) enabled are denoted by S_{det1} and S_{det2}, respectively. We enumerate these states as follows

$$\begin{aligned} S_{\text{exp}} &= \{1, 2, \ldots, N_1\}, \\ S_{\text{det1}} &= \{N_1 + 1, N_1 + 2, \ldots, N_1 + N_2\}, \\ S_{\text{det2}} &= \{N_1 + N_2 + 1, N_1 + N_2 + 2, \ldots, N\}. \end{aligned} \tag{5}$$

The transition kernel is a square matrix of dimension N and, in general, its ij-entries are functions of clock readings associated with the current state i and intervals for clock readings associated with the new state j. Due to the construction of the GSSMC $\{X(nD)\colon\ n \geq 0\}$ described in the previous section, for DSPNs with at most two deterministic transitions concurrently enabled, the transition kernel of the underlying GSSMC can be written as $\mathbf{P}(c_1, c_2, a_1, a_2)$. Subsequently, its ij-entries are defined as conditional probabilities that the next state is j with clock readings $C_1 \in [0, a_1]$ and $C_2 \in [0, a_2]$ given that the current state is i with clock readings $C_1 = c_1$ and $C_2 = c_2$.

$$\begin{aligned} p_{ij}(c_1, c_2, a_1, a_2) =& P\,\{S((n+1)D) = j,\, C_1((n+1)D) \leq a_1,\, C_2((n+1)D) \leq a_2, \\ & \mid S(nD) = i,\, C_1(nD) = c_1,\, C_2(nD) = c_2\}\,. \end{aligned} \tag{6}$$

With Eq. (6) and definition (5), the general form of the transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ for the GSSMC underlying a DSPN in which all deterministic transitions have the same firing delay D and in any marking at most two deterministic transitions are concurrently enabled can be written as a composition of 9 submatrices $\mathbf{P}_{ij}(\cdot)$ of appropriate dimension for $0 \leq c_1, c_2 \leq D$ and $0 \leq a_1, a_2 \leq D$. For the case $c_1 \leq c_2$ and $a_1 \leq a_2$ the transition kernel has the form:

$$\mathbf{P}(c_1, c_2, a_1, a_2) = \left[\begin{array}{c|c|c} \mathbf{P}_{11}(a_2) & \mathbf{P}_{12}(a_1, a_2) & \mathbf{P}_{13}(a_1, a_2) \\ \hline \mathbf{P}_{21}(c_1, a_2) & \mathbf{P}_{22}(c_1, a_1, a_2) & \mathbf{P}_{23}(c_1, a_1, a_2) \\ \hline \mathbf{P}_{31}(c_1, a_2) & \mathbf{P}_{32}(c_1, c_2, a_1, a_2) & \mathbf{P}_{33}(c_1, c_2, a_1, a_2) \end{array}\right] \begin{array}{c} 1 \\ \vdots \\ N_1 \\ \hline N_1+1 \\ \vdots \\ N_1+N_2 \\ \hline N_1+N_2+1 \\ \vdots \\ N \end{array}$$

$$1 \quad N_1 \quad \Big| \quad N_1+1 \quad N_1+N_2 \quad \Big| \quad N_1+N_2+1 \quad N$$

For $c_1 \geq c_2$ and $a_1 \leq a_2$ the transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ is of similar form. The only difference lies in that submatrix $\mathbf{P}_{31}(\cdot)$ may depend on c_2 instead of c_1, i.e., for $c_1 \geq c_2$ and $a_1 \leq a_2$ this submatrix is of the form $\mathbf{P}_{31}(c_2, a_2)$. It is important to observe that the transition kernel is symmetric with respect to $c_1 \le c_2$ and $c_1 \ge c_2$. Thus, it is sufficient to compute the kernel matrix just for the former case. Note that for a large number of applications, most of the ij-entries of the transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ are non-negative real numbers, i.e., they do not depend on the remaining firing times c_1 and/or c_2. In the special case that all ij-entries of $\mathbf{P}(c_1, c_2, a_1, a_2)$ are independent from c_1 and c_2, the transition kernel can be considered as a probability matrix for a discrete-time Markov chain for any fixed value of a_1 and a_2 (see e.g., the analysis of the M/D/2 queueing system [8]).

The transition kernel of the GSSMC underlying a DSPN in which deterministic transitions of the DSPN have different firing delays, say $D_1, D_2, \ldots, D_M$, and in any marking at most two deterministic transitions are concurrently enabled is also of the form $\mathbf{P}(c_1, c_2, a_1, a_2)$ and can be written as composition of 9 submatrices as shown above. However, for this case using observation (iv) and (v) a state i in which a deterministic transition with firing delay $D_{m(i)} > D$ is enabled (thus, the remaining firing time lies in the interval $(0, D_{m(i)}]$) is split into $\alpha_{m(i)}$ states, denoted by (i, k), with $k = 0, 1, \ldots, \alpha_{m(i)} - 1$, such that the remaining firing time of the deterministic transition lies in state (i, k) in the interval $(kD, (k+1)D]$ and in a state $(i, \alpha_{m(i)})$ in which the remaining firing time lies in the interval $(\alpha_{m(i)}D, \alpha_{m(i)}D + \gamma_{m(i)}]$. Due to this splitting of such states i and j in which one deterministic transition with firing delay $D_{m(i)}$ and $D_{m(j)}$ are enabled, respectively, each ij-entry $p_{ij}(c_1, a_1, a_2)$ of the submatrix $\mathbf{P}_{22}(c_1, a_1, a_2)$ of the transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ results in a rectangular matrix of dimension $\alpha_{m(i)} + 1 \times \alpha_{m(j)} + 1$. Accordingly, ij-entries of other submatrices of the transition kernel result also in vectors or rectangular matrices.

Using observation (iii) to (v), we distinguish the following three cases for ij-entries of the submatrix $\mathbf{P}_{22}(c_1, a_1, a_2)$ of the transition kernel:

(1) the deterministic transition fires in the interval $(nD, (n+1)D]$. Then, corresponding ij-entries of the transition kernel are of the form

$$p_{(i,0)(j,\alpha_{m(j)}-1)}(c_1, a_1, a_2) \quad \text{for } 0 < c_1 \le \delta_{m(j)},$$

$$p_{(i,0)(j,\alpha_{m(j)})}(c_1, a_1, a_2) \quad \text{for } \delta_{m(j)} < c_1 \le D.$$

(2) the deterministic transition becomes preempted in the interval $(nD, (n+1)D]$. Then, corresponding ij-entries of the transition kernel are of the form

$$p_{(i,k)(j,\alpha_{m(j)}-1)}(c_1, a_1, a_2) \quad \text{for } 0 < c_1 \le \delta_{m(j)} \text{ and } 1 \le k \le \alpha_{m(i)},$$

$$p_{(i,k)(j,\alpha_{m(j)})}(c_1, a_1, a_2) \quad \text{for } \delta_{m(j)} < c_1 \le D \text{ and } 1 \le k \le \alpha_{m(i)}.$$

(3) the deterministic transition does not fire in the interval $(nD, (n+1)D]$ and is still enabled at time $(n+1)D$. Then, corresponding ij-entries of the transition kernel are of the form

$$p_{(i,k)(j,k-1)}(c_1, a_1, a_2) \quad \text{for } 0 < c_1 \le D \text{ and } 1 \le k \le \alpha_{m(i)}.$$

In [14], the concept of *subordinated Markov chains* for the efficient algorithmic computation of the probability matrix of the embedded discrete-time Markov chain underlying a DSPN without concurrent deterministic transitions has been introduced. A subordinated Markov chain of a deterministic transition is a continuous-time Markov chain whose generator matrix is given by firing rates of exponential transitions concurrently and/or competitively enabled with this deterministic transition and corresponding compound transition probabilities corresponding to probabilities for firing sequences of immediate transitions.

(1) Construction of the tangible reachability graph of the DSPN which defines state transition graph of the GSMP. Derivation of generator matrices for continuous-time Markov chains subordinated to each state i $(1 \leq i \leq N)$.
(2) Numerical computation of the ij-entries of the transition kernel by transient analysis of subordinated Markov chains and subsequent summation of appropriately selected transient state probabilities (see [16] for a detailed description).
(3) Derivation of the stationary equations of the GSSMC which constitutes a system of Volterra integral equations of the second type. This step is described in Section 3.
(4) Numerical solution of the system of Volterra integral equations by using a Volterra–Runge–Kutta method or by employing an appropriate (multi-dimensional) Gauss quadrature formula (see [3] for details on such numerical methods).
(5) Computation of the stationary probabilities of the marking process of the DSPN from the solution of the Volterra integral equations using the normalization condition (see Eqs. (11) and (21)).

Fig. 1. High-level description of the proposed numerical algorithm.

In the same way, we can now define a subordinated Markov chain for each tangible marking (state) of the DSPN, i.e., also for states in which only exponential transitions are enabled. As shown in [16], due to observations (i) to (v), ij-entries of the transition kernel which depend on c_1 and/or c_2 can be expressed as appropriate sums of transient state probabilities of the subordinated Markov chains for any fixed real values for c_1 and c_2 with $0 < c_1, c_2 \leq D$. Thus, numerical computation of ij-entries of the transition kernel requires only some summations once the transient state probabilities of the subordinated Markov chains have been obtained, irrespective of the number of deterministic transitions enabled in states i and j. On the other hand, computation of the transient state probabilities themselves requires a vector-matrix multiplication at each iteration of the randomization technique [12]. As a consequence, the number of deterministic transitions concurrently enabled in tangible markings of the DSPN hardly influences the effort required for computation of ij-entries of the transition kernel. Furthermore, this effort can be reduced for many DSPNs by exploiting isomorphisms and special structure of subordinated Markov chains as proposed for the transient analysis of subordinated Markov chains of a DSPN without concurrent deterministic transitions [15].

2.4. Algorithmic description

Figure 1 summarizes the steps of the proposed numerical method for computing steady-state distributions of DSPNs with concurrently enabled deterministic transitions. The computational complexity of the method arises mainly from numerical computation of the ij-entries of the transition kernel in step (2) and solution of the system of Volterra integral equations in step (4). Note that the generation of the tangible reachability graph of the DSPN and the construction of the GSMP underlying the DSPN in step (1) requires the same asymptotical effort as for DSPNs without concurrent deterministic transitions. The computational effort of steps (3) and (5) are negligible since they require only a constant number of operations for each tangible marking of the DSPN.

3. The system of stationary equations

Using (3) and (5), we define stationary probabilities π_i for $i \in S_{\text{exp}}$, $\pi_i(a_1)$ for $i \in S_{\text{det1}}$, and $\pi(a_1, a_2)$ for $i \in S_{\text{det2}}$ of the GSSMC underlying a DSPN in which two deterministic transitions may be concurrently enabled.

$$\begin{aligned}
\pi_i &= \lim_{n\to\infty} P\{S(nD)=i\} = \lim_{t\to\infty} P\{S(t)=i\} && \text{for } 1 \le i \le N_1,\\
\pi_i(a_1) &= \lim_{n\to\infty} P\{S(nD)=i, C_1(nD)\le a_1\}\\
&= \lim_{t\to\infty} P\{S(t)=i, C_1(t)\le a_1\} && \text{for } N_1+1 \le i \le N_1+N_2,\\
\pi_i(a_1,a_2) &= \lim_{n\to\infty} P\{S(nD)=i, C_1(nD)\le a_1, C_2(nD)\le a_2\}\\
&= \lim_{t\to\infty} P\{S(t)=i, C_1(t)\le a_1, C_2(t)\le a_2\} && \text{for } N_1+N_2+1 \le i \le N.
\end{aligned}$$

From observation (i) and (ii) together with the definition of a stationary measure of a GSSMC (see e.g., [11]) follows that for a DSPN, in which all deterministic transitions have the same firing delay D, the stationary equations of the underlying embedded chain $\{X(nD):\ n \ge 0\}$ are of the form

$$\begin{aligned}
\pi_i = &\sum_{j=1}^{N_1} \pi_j \cdot p_{ji}(a_2) + \sum_{j=N_1+1}^{N_1+N_2} \int_0^{a_1} \frac{d\pi_j(c_1)}{dc_1} \cdot p_{ji}(c_1,a_2)\,dc_1\\
&+ 2\cdot \sum_{j=N_1+N_2+1}^{N} \int_0^{a_1}\int_0^{c_2} \frac{\partial^2 \pi_j(c_1,c_2)}{\partial c_1 \partial c_2} \cdot p_{ji}(c_1,a_2)\,dc_1\,dc_2
\end{aligned} \tag{7}$$

for $1 \le i \le N_1$ and $0 < a_1, a_2 \le D$ with $c_1 \le c_2$ and $a_1 = a_2$.

$$\begin{aligned}
\pi_i(a_1) = &\sum_{j=1}^{N_1} \pi_j \cdot p_{ji}(a_1, a2) + \sum_{j=N_1+1}^{N_1+N_2} \int_0^{a_1} \frac{d\pi_j(c_1)}{dc_1} \cdot p_{ji}(c_1,a_1,a_2)\,dc_1\\
&+ 2\cdot \sum_{j=N_1+N_2+1}^{N} \int_0^{a_1}\int_0^{c_2} \frac{\partial^2 \pi_j(c_1,c_2)}{\partial c_1 \partial c_2} \cdot p_{ji}(c_1,c_2,a_1,a_2)\,dc_1\,dc_2
\end{aligned} \tag{8}$$

for $N_1+1 \le i \le N_1+N_2$ and $0 < a_1, a_2 \le D$ with $c_1 \le c_2$ and $a_1 = a_2$.

$$\begin{aligned}
\pi_i(a_1,a_2) = &\sum_{j=1}^{N_1} \pi_j \cdot p_{ji}(a_1, a2) + \sum_{j=N_1+1}^{N_1+N_2} \int_0^{a_1} \frac{d\pi_j(c_1)}{dc_1} \cdot p_{ji}(c_1,a_1,a_2)\,dc_1\\
&+ 2\cdot \sum_{j=N_1+N_2+1}^{N} \int_0^{a_1}\int_0^{c_2} \frac{\partial^2 \pi_j(c_1,c_2)}{\partial c_1 \partial c_2} \cdot p_{ji}(c_1,c_2,a_1,a_2)\,dc_1\,dc_2\\
&+ \sum_{j=N_1+N_2+1}^{N} \int_{a_1}^{a_2}\int_0^{a_1} \frac{\partial^2 \pi_j(c_1,c_2)}{\partial c_1 \partial c_2} \cdot \tilde{p}_{ji}(c_2,a_1,a_2)\,dc_1\,dc_2
\end{aligned} \tag{9}$$

for $N_1+N_2+1 \le i \le N$ and $0 < a_1, a_2 \le D$ with $c_1 \le c_2$ and $a_1 \le a_2$.

In Eq. (9), expressions of the form $\tilde{p}_{ji}(c_2, a_1, a_2)$ are derived from the kernel elements $p_{ji}(c_1, c_2, a_1, a_2)$ by projection from the two-dimensional plane $[0, a_1] \times [0, a_2]$ on the one-dimensional line $[a_1, a_2]$. Equations (7) to (9), in general, comprise a system of integro-differential equations with constant boundary conditions

$$\begin{aligned}
\pi_i(0) &= 0 \quad \text{for } N_1+1 \le i \le N_1+N_2,\\
\pi_i(c_1,0) &= 0 \quad \text{for } N_1+N_2+1 \le i \le N \text{ and } 0 \le c_1 \le D,
\end{aligned}$$

$$\pi_i(0, c_2) = 0 \quad \text{for } N_1 + N_2 + 1 \le i \le N \text{ and } 0 \le c_2 \le D. \tag{10}$$

The system of integro-differential Eqs. (7) to (9) together with the boundary conditions (10) and the normalization Eq. (11) uniquely determine the stationary probabilities of the marking process of a DSPN with markings enabling two deterministic transitions with firing delay D, concurrently.

$$\sum_{j=1}^{N_1} \pi_j + \sum_{j=N_1+1}^{N_1+N_2} \pi_j(D) + \sum_{j=N_1+N_2+1}^{N} \pi_j(D, D) = 1. \tag{11}$$

This system of integro-differential equations can be transformed, using integration by parts and the boundary conditions into a system of Volterra integral equations of the second type. Using integration by parts, single integrals with a first derivative of a stationary probability are transformed into a sum of two terms. For example, we have

$$\int_0^{a_1} \frac{d\pi_j(c_1)}{dc_1} \cdot p_{ji}(c_1, a_1, a_2)\, dc_1 = \pi_j(a_1) p_{ji}(c_1, a_1, a_2)\big|_{c_1=a_1} - \int_0^{a_1} \pi_j(c_1) \frac{dp_{ji}(c_1, a_1, a_2)}{dc_1} dc_1. \tag{12}$$

With symbolic integration with respect to c_1 and with integration by parts, a rectangular double integral with a second partial derivative is transformed into an expression which contains two products of the stationary probability with kernel elements and a single integral of the product of the stationary probability and the corresponding partial derivative of the kernel element. Thus, we have

$$\begin{aligned} \int_{a_1}^{a_2} \int_0^{a_1} \frac{\partial^2 \pi_j(c_1, c_2)}{\partial c_1 \partial c_2} \cdot \tilde{p}_{ji}(c_2, a_1, a_2)\, dc_1\, dc_2 &= \pi_2(a_1, a_2) \cdot \tilde{p}_{ji}(c_2, a_1, a_2)\big|_{c_2=a_2} \\ &- \pi_2(a_1, a_1) \cdot \tilde{p}_{ji}(c_2, a_1, a_2)\big|_{c_2=a_1} - \int_{a_1}^{a_2} \pi_2(a_1, a_2) \frac{\partial \tilde{p}_{ji}(c_2, a_1, a_2)}{\partial c_2} dc_2. \end{aligned} \tag{13}$$

A triangular double integral with a second partial derivative can be transformed into a rectangular integral using the substitution $c_1 := c_1 c_2 / a_1$ and, then, integration by parts can be applied twice. Thus, we can replace all (partial) derivatives of stationary probabilities in the system of Eqs. (7) to (9) by (partial) derivatives of kernel elements. Since the former are unknown functions, whereas the latter can be numerically determined using the Chapman–Kolmogorov equation for continuous-time Markov chains with discrete state space (see [16] for details), this transformation simplifies the numerical solution of the stationary equations, considerably. In fact, besides the numerical computation of kernel elements and their derivatives, we only have to solve a system of Volterra integral equations of the second type, for which standard solution methods are available [3]. From formulas (7) to (13), it should be clear that the proposed approach for deriving stationary equations of the GSSMC can be extended in straight-forward manner to the case of DSPNs in which $L \ge 2$ deterministic transitions are concurrently enabled. The resulting system of Volterra integral equations then comprises terms with L-dimensional integrals.

Numerical solution methods for Volterra equations are based an appropriate discretization scheme for integral expression and solution of a linear system of equations for each mesh point. Thus, the main memory requirements of the proposed method are of the same order as for numerical steady state analysis of DSPNs without concurrent deterministic transitions [14] whereas the computational effort is higher, since not just one linear system of equations needs to be solved. However, the additional effort is still reasonable; in particular, the overall effort for solving the Volterra equations is not given by the product of the number of mesh points and the effort of solving the first linear system. This is because for solution of subsequent linear systems considerably fewer iterations are required than for solution of the first linear system, since the solution of the linear system for the previous step is chosen as start vector for the iterative solver.

To deal with deterministic transitions with different firing delays, consider a state i in which one deterministic transition with delay $D_{m(i)}$ is enabled. Using observations (iii) to (v), we split the stationary probability $\pi_i(a_1)$ with $0 < a_1 \le D_{m(i)}$ into $2\alpha_{m(i)} + 1$ stationary probabilities

$$\pi^-_{i,k}(a_1) \quad \text{for } k = 0, 1, \ldots, \alpha_{m(i)} \quad \text{with } kD < a_1 \le kD + \gamma_{m(i)}, \tag{14}$$

$$\pi^+_{i,k}(a_1) \quad \text{for } k = 0, 1, \ldots, \alpha_{m(i)} - 1 \quad \text{with } kD + \gamma_{m(i)} < a_1 \le (k+1)D. \tag{15}$$

When at most one deterministic transition may be enabled in any marking of the DSPN, the stationary equations of the GSSMC $\{X(nD): n \ge 0\}$ are of the form

$$\begin{aligned}\pi_i = {} & \sum_{j=1}^{N_1} \pi_j \cdot p_{ji}(a_1) + \sum_{j=N_1+1}^{N_1} 1_{(a_1 \le \gamma_{m(j)})} \sum_{k=0}^{\alpha_{m(j)}} \int_0^{a_1} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i)}(c_1, a_1)\, dc_1 \\ & + \sum_{j=N_1+1}^{N} 1_{(a_1 > \gamma_{m(j)})} \left(\sum_{k=0}^{\alpha_{m(j)}} \int_0^{\gamma_{m(j)}} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i)}(c_1, a_1)\, dc_1 \right. \\ & \left. + \sum_{k=0}^{\alpha_{m(j)}-1} \int_{\gamma_{m(j)}}^{a_1} \frac{d\pi^+_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i)}(c_1 + \gamma_{m(j)}, a_1)\, dc_1 \right)\end{aligned} \tag{16}$$

for $1 \le i \le N_1$ and $0 < a_1 \le D$

$$\begin{aligned}\pi^-_{i,\alpha_{m(i)}}(a_1) = {} & \sum_{j=1}^{N_1} \pi_j \cdot p_{(j)(i,\alpha_{m(i)})}(a_1) + \sum_{j=N_1+1}^{N_1} 1_{(a_1 \le \gamma_{m(j)})} \sum_{k=0}^{\alpha_{m(j)}} \int_0^{a_1} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)})}(c_1, a_1)\, dc_1 \\ & + \sum_{j=N_1+1}^{N} 1_{(a_1 > \gamma_{m(j)})} \left(\sum_{k=0}^{\alpha_{m(j)}} \int_0^{\gamma_{m(j)}} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)})}(c_1, a_1)\, dc_1 \right. \\ & \left. + \sum_{k=0}^{\alpha_{m(j)}-1} \int_{\gamma_{m(j)}}^{a_1} \frac{d\pi^+_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)})}(c_1 + \gamma_{m(j)}, a_1)\, dc_1 \right)\end{aligned} \tag{17}$$

for $N_1 + 1 \le i \le N$ and $0 < a_1 \le \gamma_{m(i)}$.

$$\begin{aligned}\pi^+_{i,\alpha_{m(i)}-1}(a_1) = {} & \sum_{j=1}^{N_1} \pi_j \cdot p_{(j)(i,\alpha_{m(i)}-1)}(a_1) \\ & + \sum_{j=N_1+1}^{N_1} 1_{(a_1 \le \gamma_{m(j)})} \sum_{k=0}^{\alpha_{m(j)}} \int_0^{a_1} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)}-1)}(c_1, a_1)\, dc_1 \\ & + \sum_{j=N_1+1}^{N} 1_{(a_1 > \gamma_{m(j)})} \left(\sum_{k=0}^{\alpha_{m(j)}} \int_0^{\gamma_{m(j)}} \frac{d\pi^-_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)}-1)}(c_1, a_1)\, dc_1 \right. \\ & \left. + \sum_{k=0}^{\alpha_{m(j)}-1} \int_{\gamma_{m(j)}}^{a_1} \frac{d\pi^+_{j,k}(c_1)}{dc_1} \cdot p_{(j,k)(i,\alpha_{m(i)}-1)}(c_1 + \gamma_{m(j)}, a_1)\, dc_1 \right)\end{aligned} \tag{18}$$

for $N_1 + 1 \le i \le N$ and $\gamma_{m(i)} < a_1 \le D$.

Additional equations are needed to uniquely determine the stationary probabilities $\pi_{i,k}^{-}(a_1)$ and $\pi_{i,k}^{+}(a_1)$. These equations are of the form

$$\pi_{i,k-1}^{-}(a_1) = \sum_{j=N_1+1}^{N} \int_0^{a_1} \frac{d\pi_{j,k}^{-}(c_1)}{dc_1} p_{(j,k)(i,k-1)}(c_1, a_1)\, dc_1 \tag{19}$$

for $0 < a_1 \le \gamma_{m(i)}$ and $k = 1, 2, \ldots, \alpha_{m(i)}$.

$$\pi_{i,k-1}^{+}(a_1) = \sum_{j=N_1+1}^{N} \int_{\gamma_{m(i)}}^{a_1} \frac{d\pi_{i,k}^{+}(c_1)}{dc_1} p_{(j,k)(i,k-1)}(c_1 + \gamma_{m(i)}, a_1)\, dc_1 \tag{20}$$

for $\gamma_{m(i)} < a_1 \le D$ and $k = 1, 2, \ldots, \alpha_{m(i)} - 1$.

The normalization equation is

$$\sum_{j=1}^{N_1} \pi_j + \sum_{j=N_1+1}^{N} \left(\sum_{k=0}^{\alpha_{m(j)}-1} \left(\pi_{j,k}^{+}(D) + \pi_{j,k}^{-}(\gamma_{m(j)}) \right) + \pi_{j,\alpha_{m(j)}}^{-}(\gamma_{m(j)}) \right) = 1. \tag{21}$$

The stationary equations comprise a system of integro-differential equations of the same form as when all deterministic transitions have the same firing delay. The system of equations has the constant boundary conditions

$$\begin{aligned} \pi_{i,k}^{+}(\gamma_{m(i)}) &= 0 \quad \text{for } 0 \le k \le \alpha_{m(i)} - 1 \quad \text{and } N_1 + 1 \le i \le N, \\ \pi_{i,k}^{-}(0) &= 0 \quad \text{for } 0 \le k \le \alpha_{m(i)} \qquad \text{and } N_1 + 1 \le i \le N. \end{aligned} \tag{22}$$

Using integration by parts as discussed above, we obtain a system of Volterra equations of the second type which can be solved numerically with the same techniques as the system of Eqs. (7) to (10). Subsequently, the stationary probability of the marking process underlying the DSPN for a state i, in which a deterministic transition with firing delay $D_{m(i)}$ is enabled, is given by

$$\pi_i(D_{m(i)}) = \sum_{k=0}^{\alpha_{m(i)}-1} \left(\pi_{i,k}^{+}(D) + \pi_{i,k}^{-}(\gamma_{m(i)}) \right) + \pi_{i,\alpha_{m(i)}}^{-}(\gamma_{m(i)}). \tag{23}$$

Recall that if $D_{m(i)} = D$, then $\alpha_{m(i)} = 1$, $\delta_{m(i)} = D$, and $\gamma_{m(i)} = 0$ for a state i. In this case, stationary Eqs. (10) and (11) reduce to equations of the form (1) and (2), but without the double integral expressions. Equation (12) is obsolete since $\gamma_{m(i)} = 0$. From Eqs. (17) and (18), it should be clear how the stationary equations of the GSSMC are derived for a DSPN in which several deterministic transitions (with possibly different firing delays) are concurrently enabled.

4. An example: The M/D/2/*K* queueing system

To illustrate the derivation of the transition kernel for a GSSMC underlying a DSPN with concurrently enabled deterministic transitions, we consider a DSPN of a M/D/2/K queueing system. Figure 2 shows a DSPN for this queueing system. The K tokens residing in place p_8 in the initial marking represent the finite number of buffers of the queueing system. All arcs of the DSPN have multiplicity 1. The exponential transition t_3 has the mean firing delay $1/\lambda$ and represents the Poisson arrival stream of customers. Tokens contained in place p_1 represent customers waiting in the queue. Tokens contained

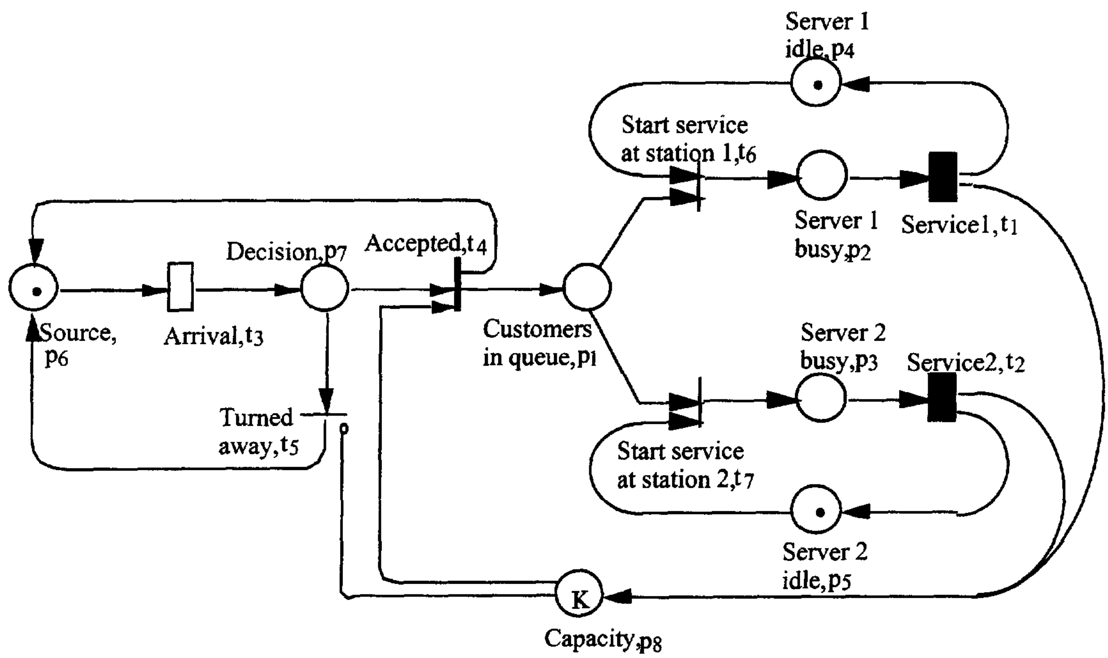

Fig. 2. DSPN of the M/D/2/K queue with customers turned away.

in the places p_2 and p_3 represent customers currently being served. The constant service requirements are modeled by the deterministic transitions t_1 and t_2 which have firing delay D. For ease of exposition, we assume that the immediate transitions t_6 and t_7 have both associated the firing weights 1/2, such that arriving customers to an empty system join each server with equal probability. The DSPN of the M/D/2/K queue with customers turned away is of particular interest because complex DSPNs often contain subnets which are equivalent to this DSPN. Tangible markings of this DSPN can be uniquely specified by the numbers of tokens residing in places p_1, p_2, and p_3. Defining the state number $s = \#p_1 + \#p_2 + 2 * \#p_3 + 1$, the set of tangible markings (state space of the GSMP) can be written as $S = \{1, 2, \dots, K + 2\}$ where using (5) we decompose the set S in

$$S_{\text{exp}} = \{1\}, \qquad S_{\text{det1}} = \{2, 3\}, \quad \text{and} \quad S_{\text{det2}} = \{4, 5, \dots, K + 2\}.$$

Thus, we have $N_1 = 1$, $N_2 = 2$, and $N = K + 2$. Figure 3 shows the tangible reachability graph of the DSPN of Fig. 2 which also defines the state transition graph for the corresponding GSMP. The transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ of the GSSMC contains the following three different types of entries:

(1) Constant entries given by Poisson probabilities which are independent from the remaining firing times for deterministic transitions t_1 and t_2.

$$b_i = P\{i \text{ arrivals in } (0, D]\} = \frac{(\lambda D)^i}{i!} \cdot \mathrm{e}^{-\lambda D} \tag{24}$$

for $i = 0, 1, \dots, K - 2$.

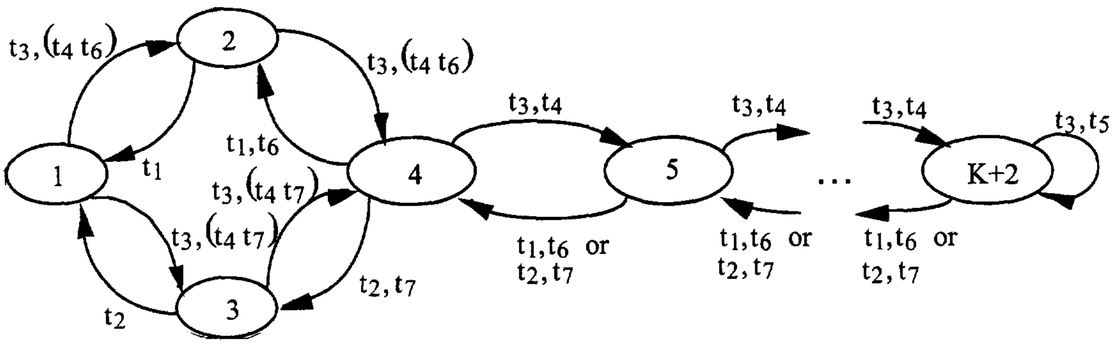

Fig. 3. Tangible reachability graph of the M/D/2/K queue with customers turned away.

$$B_i = P\{\text{at least } i \text{ arrivals in } (0, D]\} = 1 - \sum_{k=0}^{i-1} b_k \tag{25}$$

for $i = 2, 3, \ldots, K-1$.

(2) Functional entries setting remaining firing times for deterministic transitions t_1 and t_2 (corresponding to intervals for new clock values a_1 and a_2 in the GSSMC). For the case $c_1 \le c_2$ and $a_1 \le a_2$ these functional entries are given by

$$\phi_0(a_2) = P\{\text{no arrivals in } (0, a_2]\} = \mathrm{e}^{-\lambda a_2}, \tag{26}$$

$$\begin{aligned}\phi_1(a_1, a_2) &= P\{\text{one arrival in } (0, a_1] \text{ and no arrivals in } (a_1, a_2]\}\\ &= \lambda a_1 \mathrm{e}^{-\lambda a_2},\end{aligned} \tag{27}$$

$$\begin{aligned}\phi_1(a_1 - c_1, a_2) &= P\{\text{no arrivals in } (0, c_1] \text{ and one arrival in } (c_1, a_1]\\ &\qquad \text{and no arrivals in } (a_1, a_2]\}\\ &= \lambda a_1 \mathrm{e}^{-\lambda a_2} - \lambda c_1 \mathrm{e}^{-\lambda a_2},\end{aligned} \tag{28}$$

$$\begin{aligned}\phi_1(a_1 + c_1, a_2) &= P\{\text{one arrival in } (0, c_1] \text{ and no arrivals in } (c_1, a_2]\}\\ &\quad + P\{\text{one arrival in } (0, a_1] \text{ and no arrivals in } (a_1, a_2]\}\\ &= \lambda a_1 \mathrm{e}^{-\lambda a_2} + \lambda c_1 \mathrm{e}^{-\lambda a_2},\end{aligned} \tag{29}$$

$$\begin{aligned}\phi_i(a_1, a_2) &= P\{i \text{ arrivals in } (0, D] \text{ where at least one in } (0, a_1] \text{ and at least one in } (0, a_2]\}\\ &= \frac{(\lambda D)^i}{i!} \cdot \mathrm{e}^{-\lambda D} - \frac{(\lambda(D - a_1))^i}{i!} \cdot \mathrm{e}^{-\lambda D} - \frac{\lambda^i a_1 (D - a_2)^{i-1}}{(i-1)!} \cdot \mathrm{e}^{-\lambda D}\end{aligned} \tag{30}$$

for $i = 2, 3, \ldots, K-1$ and $0 < a_1, a_2 \le D$.

$$\begin{aligned}\Phi_K(a_1, a_2) &= P\{\text{at least } K \text{ arrivals in } (0, D] \text{ where at least one in } (0, a_1] \text{ and at least one in } (0, a_2]\}\\ &= 1 - \mathrm{e}^{-\lambda a_1} - \lambda a_1 \mathrm{e}^{-\lambda a_2} - \mathrm{e}^{-\lambda D} \sum_{k=2}^{K-1} \frac{(\lambda D)^k}{k!} - \frac{(\lambda(D - a_1))^k}{k!} - \frac{\lambda^k a_1 (D - a_2)^{k-1}}{(k-1)!}\end{aligned} \tag{31}$$

for $0 < a_1, a_2 \le D$.

(3) Functions taking into consideration remaining firing times of deterministic transitions t_1 and t_2 (corresponding to old clock values c_1 and c_2 in the GSSMC). For the case $c_1 \le c_2$ these functions are

given by

$$\Psi_i(c_1) = P\{\text{at least } i \text{ arrivals in } (0, c_2] \text{ and no arrival in } (c_1, D]\}$$
$$= e^{-\lambda(D-c_1)} - e^{-\lambda D} \sum_{k=0}^{i-1} \frac{(\lambda c_1)^k}{k!} \tag{32}$$

for $i = 1, 2, \ldots, K$ and $0 < c_1 \leq D$.

$$\varphi_i(c_1, c_2) = P\{\text{at least } i \text{ arrivals in } (0, c_2] \text{ where at least one arrival in } (c_1, c_2] \text{ and no arrivals in } (c_2, D]\}$$
$$= e^{-\lambda(D-c_2)} \sum_{k=0}^{i-2} \frac{(\lambda c_1)^k}{k!} e^{-\lambda c_1} - \frac{(\lambda c_2)^k}{k!} e^{-\lambda c_2} \tag{33}$$

for $i = 2, 3, \ldots, K$ and $0 < c_1, c_2 \leq D$.

Recall from Section 2.3 that the transition kernel of the GSSMC underlying a DSPN in which two deterministic transition transitions with the same firing delay D are concurrently enabled can be written as a composition of 9 submatrices $\mathbf{P}_{ij}(\cdot)$ with $1 \leq i, j \leq 3$. Using the constant entries given in (24) and (25) and functional entries of (26) to (33), for the case $c_1 \leq c_2$ these 9 submatrices of the transition kernel $\mathbf{P}(c_1, c_2, a_1, a_2)$ of the GSSMC underlying the DSPN of Fig. 2 are given by

$$\mathbf{P}_{11}(a_2) = [\phi_0(a_2)],$$

$$\mathbf{P}_{12}(a_1, a_2) = \begin{bmatrix} \frac{1}{2}\phi_1(a_1, a_2) & \frac{1}{2}\phi_1(a_1, a_2) \end{bmatrix},$$

$$\mathbf{P}_{13}(a_1, a_2) = \begin{bmatrix} \phi_2(a_1, a_2) & \phi_3(a_1, a_2) & \cdots & \phi_{K-1}(a_1, a_2) & \Phi_K(a_1, a_2) \end{bmatrix},$$

$$\mathbf{P}_{21}(a_2) = \begin{bmatrix} \phi_0(a_2) \\ \phi_0(a_2) \end{bmatrix},$$

$$\mathbf{P}_{22}(c_1, a_1, a_2) = \begin{bmatrix} \frac{1}{2}\phi_1(a_1 - c_1, a_2) & \frac{1}{2}\phi_1(a_1 + c_1, a_2) \\ \frac{1}{2}\phi_1(a_1 + c_1, a_2) & \frac{1}{2}\phi_1(a_1 - c_1, a_2) \end{bmatrix},$$

$$\mathbf{P}_{23}(c_1, a_1, a_2) = \begin{bmatrix} \phi_2(a_1, a_2) & \phi_3(a_1, a_2) & \cdots & \phi_{K-1}(a_1, a_2) + \psi_K(c_1) & \Phi_K(a_1, a_2) - \psi_K(c_1) \\ \phi_2(a_1, a_2) & \phi_3(a_1, a_2) & \cdots & \phi_{K-1}(a_1, a_2) + \psi_K(c_1) & \Phi_K(a_1, a_2) - \psi_K(c_1) \end{bmatrix},$$

$$\mathbf{P}_{31}(a_2) = \begin{bmatrix} \phi_0(a_2) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

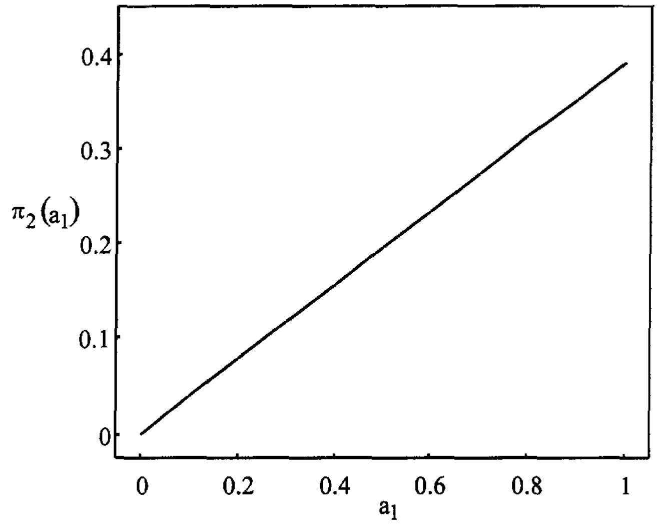

Fig. 4. M/D/2/2 queue: The probability distribution function $\pi_2(a_1)$.

$$\mathbf{P}_{32}(c_1, c_2, a_1, a_2) = \begin{bmatrix} \frac{1}{2}\phi_1(a_1 + (c_2 - c_1), a_2) & \frac{1}{2}\phi_1(a_1 - (c_2 - c_1), a_2) \\ \frac{1}{2}b_0 & \frac{1}{2}b_0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{P}_{33}(c_1, c_2, a_1, a_2) =$$

$$\begin{bmatrix} \phi_2(a_1, a_2) & \cdots & \cdots & \phi_{K-2}(a_1, a_2) + \psi_{K-1}(c_1) & \phi_{K-1}(a_1, a_2) + \varphi_K(c_1, c_2) & \Phi_K(a_1, a_2) - \psi_{K-1}(c_1) - \varphi_K(c_1, c_2) \\ b_1 & \cdots & \cdots & b_{K-3} + \psi_{K-2}(c_1) & b_{K-2} + \varphi_{K-1}(c_1, c_2) & B_{K-1} - \psi_{K-2}(c_1) - \varphi_{K-1}(c_1, c_2) \\ b_0 & b_1 & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \vdots & \vdots & \vdots \\ \vdots & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_0 + \psi_1(c_1) & b_1 + \varphi_2(c_1, c_2) & B_2 - \psi_1(c_1) - \varphi_2(c_1, c_2) \end{bmatrix}.$$

Figures 4 and 5 plot the degenerated probability functions $\pi_2(a_1)$ and $\pi_4(a_1, a_2)$ for $0 \le a_1 \le D$ and $0 \le a_1, a_2 \le D$ with $\lambda = 0.9$, $D = 1$, and $K = 2$. Intentionally, we have chosen $K = 2$ because for this case with $a_1 = a_2 = D$, analytical expressions for the stationary probabilities are known using Erlang's loss formula. These stationary probabilities are given by

$$\pi_1 = \frac{2}{2 + 2\lambda D + \lambda^2 D^2}, \qquad \pi_2(D) = \pi_3(D) = \frac{\lambda D}{2 + 2\lambda D + \lambda^2 D^2},$$

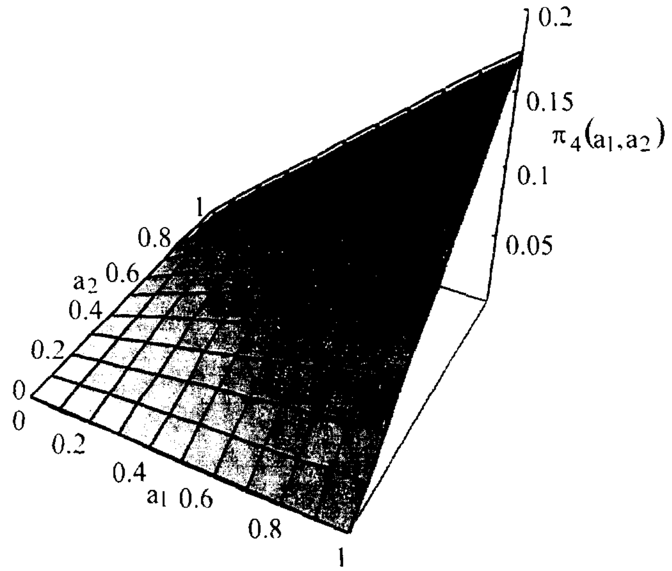

Fig. 5. M/D/2/2 queue: The probability distribution function $\pi_4(a_1, a_2)$.

$$\pi_4(D, D) = \frac{\lambda^2 D^2}{2 + 2\lambda D + \lambda^2 D^2}.$$

The curves have been obtained by numerically solving the system of Volterra integral Eqs. (7) to (9) using the transition kernel presented above and the corresponding normalization condition. To numerically solve the Volterra equations, an adaptive Simpson rule is employed for approximating both one-dimensional and two-dimensional integrals (see [3] for details) and subsequently a linear system of equations is solved for each mesh point.

Figures 4 and 5 indicate that the stationary probabilities of the GSSMC, $\pi_2(a_1)$, $\pi_3(a_1)$ and $\pi_4(a_1, a_2)$ are given by

$$\pi_2(a_1) = \pi_3(a_1) = \frac{\lambda a_1}{2 + 2\lambda D + \lambda^2 D^2}, \qquad \pi_4(a_1, a_2) = \frac{\lambda^2 a_1 a_2}{2 + 2\lambda D + \lambda^2 D^2}.$$

With some calculations using the transition kernel presented above and the system of Eqs. (7) to (9), one can verify that these expressions are indeed the symbolic solution in the continuous state space of the GSSMC. Noting that the derivatives of $\pi_2(a_1)$, $\pi_3(a_1)$ and $\pi_4(a_1, a_2)$ with respect to a_1 and a_1 and a_2, respectively, are constant, the GSSMC of the M/D/2/2 queue can be simplified to a discrete-time Markov chain with discrete state space for any fixed a_1 and a_2. Using this Markov chain with $a_1 = a_2 = D$, we can show in a new way that the steady-state distribution of the number of customers in an M/D/2/2 queue is equal to the corresponding steady-state distribution in an M/M/2/2 queue.

5. Conclusions

This paper introduced an efficient numerical method for computing stationary distributions of DSPNs with concurrently enabled deterministic transitions based on the analysis of a discrete-time embedded general state space Markov chain (GSSMC), rather than analysis of a continuous-time Markov process as in the method of supplementary variables [10]. The stochastic behavior of the GSSMC is specified by a transition kernel which, as outlined in Section 2.3 and illustrated by the example in Section 4, can be numerically determined using transient analysis of appropriately defined continuous-time Markov chains. Both the stationary analysis of the GSSMC and stationary analysis of the continuous-time Markov process lead to a system of integro-differential equations. The integro-differential equations for the embedded GSSMC, however, can always be transformed into a system of Volterra integral equations of the second type for which standard numerical methods are available (see e.g., [3]) whereas solution of the integro-differential equations of the continuous-time Markov process requires the numerical solution of a system of partial differential equation.

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